Understanding the contact ratio for spur gears with some comments on ways to read a textbook

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Our textbook covers the topic of the contact ratio of meshing spur gears in less than a page, reproduced here, together with the cited figures [1, p. 632]:

“It is obviously necessary that the tooth profiles be proportioned so that a second pair of mating teeth come into contact before the first pair is out of contact. The average number of teeth in contact as the gears rotate together is the contact ratio (CR), which is calculated from the following equation,

\[
CR = \frac{\sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} - c \sin \phi}{p_b}
\] (15.9)

where

\[r_{ap}, r_{ag} = \text{addendum radii of the mating pinion and gear}\]
\[r_{bp}, r_{bg} = \text{base circle radii of the mating pinion and gear}\]

The base pitch \(p_b\) is

\[p_b = \frac{\pi d_b}{N}\] (15.10)

where \(N = \text{number of teeth and } d_b = \text{diameter of the base circle.}\) From Figure 15.7

\[d_b = d \cos \phi \quad r_b = r \cos \phi \quad \text{and} \quad p_b = p \cos \phi\] (15.11)

The base pitch is like the circular pitch except that it represents an arc of the base circle rather than an arc of the pitch circle. It is illustrated in Figure 15.4.

In general, the greater the contact ratio, the smoother and quieter the operation of the gears. A contact ratio of 2 or more means that at least two pairs of teeth are theoretically in contact at all times.”

The key big-picture elements of this quotation are the second sentence, which defines the contact ratio, and the last paragraph which explains its significance.

We could read these paragraphs and sigh and accept that we are being presented with yet another opaque equation in a text of nearly nine-hundred pages that seems to have one or more equations on each page. This attitude treats the text as an enormous
compendium of magic spells that we must somehow keep straight in our minds.

Alternatively, we can believe that the authors have presented us with the bare bones of a derivation. They have faith in us, as active readers, to flesh out that skeleton and to create a living breathing creature. We will see that the equation for $CR$ lives among just a few simple principles of geometry, trigonometry, and algebra with which we are already well familiar. With some effort, we can remove the mystery of the equation and place it confidently among our well-understood ideas. This exercise can be very satisfying–no sighs necessary.

The derivation of the contact ratio equation, fleshed out

I will list the familiar ideas needed to come to a confident acceptance of the contact ratio equation.

1. Careful drawings, carefully labeled.
2. The law of cosines.
3. Some trigonometric identities.
4. The quadratic equation.

I think you will agree that there is nothing arcane here; you have been familiar with these tools for years. How can we know in advance that these are the needed tools? I have no answer to that question other than to say that if they are near the surface of our consciousness, it may occur to us to try them.

Figure 1 extracts some details from Figure 15.8 in the text. I have included the addenda circles, the base circles, and the pressure line for a mating pinion and gear. We note that for involute gears the path of contact, $L_{ac}$, is the straight line segment from $a$ to $c$. The contact ratio will be the length of this segment divided by the “length” of a single tooth. This is why the contact ratio is called a ratio–because it is calculated as the ratio of two lengths. Nevertheless, it is also the average number of teeth in engagement between meshing gears.

If $L_{ac}$ were $5 \text{ in}$ and the “length” of a single tooth were $2 \text{ in}$, we would say that the contact ratio was 2.5 because that is how many tooth lengths would fit along the path of contact.

Involute gears move in the same way that cylinders would move if connected by an inextensible cord wrapped around the base circles as in Figure 2. Thus, when point $b$ moves in an arc around the base circle through a distance corresponding to the arc length of one tooth, $p_b$, point $a$ will move in a straight line along the path of contact exactly the same distance. This is the insight that permits us to conclude that the “length” of a single tooth in the previous paragraph should be interpreted as the base pitch, $p_b$. 
Our goal should now be clear. We need to find $L_{ac}$ and $p_b$ in terms of the known dimensions of the pinion and gear. We can then calculate the numerical value of the contact ratio.

**Finding the length of the path of contact, $L_{ac}$**

Figure 3 highlights a triangle that will allow us to calculate $L_{aP}$, which is a portion of $L_{ac}$. Can you see that the obtuse angle within this triangle is given by $\frac{\pi}{2} + \phi$? The two known sides are the radius of the addendum circle of the gear, $r_{ag}$, and the radius of the pitch circle of the gear, $r_g$. We can apply the law of cosines to this triangle:

$$r_{ag}^2 = r_g^2 + L_{aP}^2 - 2r_gL_{aP}\cos\left(\frac{\pi}{2} + \phi\right) \quad (1a)$$

Using the trigonometric identity $\cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$, we can rewrite the previous equation as

$$r_{ag}^2 = r_g^2 + L_{aP}^2 + 2r_gL_{aP}\sin\phi \quad (1b)$$

This equation can be rearranged as

$$L_{aP}^2 + 2r_g\sin\phi L_{aP} - (r_{ag}^2 - r_g^2) = 0 \quad (1c)$$

We should recognize this version as the quadratic equation for the unknown $L_{aP}$, which has the solution

$$L_{aP} = -r_g\sin\phi + \sqrt{r_g^2\sin^2\phi + r_{ag}^2 - r_g^2} \quad (1d)$$

Using yet another trigonometric identity $\sin^2\phi = 1 - \cos^2\phi$, we can rewrite the previous expression as

$$L_{aP} = -r_g\sin\phi + \sqrt{r_g^2(1 - \cos^2\phi) + r_{ag}^2 - r_g^2} \quad (1e)$$

$$= -r_g\sin\phi + \sqrt{r_g^2 - r_g^2\cos^2\phi + r_{ag}^2 - r_g^2} \quad (1f)$$

$$= -r_g\sin\phi + \sqrt{r_{ag}^2 - r_{bg}^2} \quad (1g)$$

In the last form of this equation, we have used the fact that the radius of the base circle, $r_{bg}$ is related to the radius of the pitch circle, $r_g$, by $r_{bg} = r_g\cos\phi$.

Focusing our attention on an analogous triangle involving the pinion, we can similarly derive a relationship for the portion, $L_{Pc}$ of the contact path. The result is

$$L_{Pc} = -r_p\sin\phi + \sqrt{r_{ap}^2 - r_{bp}^2} \quad (2)$$

The total length of the path of contact can now be calculated as the sum of the results from equation (1g) and equation (2).

$$L_{ac} = L_{aP} + L_{Pc} = -r_p\sin\phi - r_g\sin\phi + \sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} \quad (3a)$$

$$= -(r_p + r_g)\sin\phi + \sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} \quad (3b)$$

$$= \sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} - c\sin\phi \quad (3c)$$
In the last form of the preceding, we have used the definition of the center-to-center distance between the pinion and gear, \( c = r_p + r_g \).

**Our result for the contact ratio, \( CR \)**

Therefore, our result for the contact ratio is

\[
CR = \frac{L_{ac}}{p_b} = \frac{\sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} - c \sin \phi}{p_b}
\]  

which is identical to what is presented in our text. By working this out for ourselves, we should have much more confidence in this equation and a better sense of the meaning of the contact ratio.

**Further thoughts**

It would be a pity to get this far and not consider the implications of our result. Evidently, a large contact ratio is good. Therefore, we should ask ourselves about trends related to our equation for the contact ratio, \( CR \). I have referred previously to these types of questions as the “All else being equal, what happens if ...” questions. For example, if the center-to-center distance and gear ratio are fixed, what happens to the contact ratio as the diametral pitch is varied? This is the same as asking what happens as the tooth size is varied.

We can use our expression for contact ratio to explore this question, by recalling that for standard gears the relationship between addenda radii and pitch radii is

\[
r_a = r + a = r + \frac{1}{P}
\]  

(5)

Also, the relationship between base circle radii and pitch radii is

\[
r_b = r \cos \phi
\]  

(6)

Also, the base pitch can be expressed in terms of the diametral pitch as

\[
p_b = \frac{\pi}{P} \cos \phi
\]  

(7)

Using equations (5), (6) and (7), we can express the contact ratio as a function of diametral pitch, \( CR(P) \), with fixed values of center-to-center distance and gear ratio. The details are left as an exercise for the reader. We can then use that function to plot \( CR \) as a function of \( P \). This plot should give us a visual indication of the effect of tooth size on contact ratio. We can repeat the plot for different gear ratios to obtain a visual indication of the effect of gear ratio on contact ratio. We can also repeat the plot for different choices of pressure angle to get a visual indication of the effect of pressure angle.

We will be able to answer the following questions by looking at our plots. All else being equal, do small or large teeth lead to a larger contact ratio? All else being equal, does a gear ratio close or distant from unity lead to a larger contact ratio? All else being equal, does a small or large pressure angle lead to a larger contact ratio?

What I have described is a brute force numerical method to spot these trends.
Exercise

Let the pressure angle be $\phi = 25^\circ$. Let the center-to-center distance between pinion and gear be $c = 12$ in. Investigate four different gear ratios $\frac{z_2}{z_p} = 1, 2, 3, 5$. For each gear ratio, plot the contact ratio, $CR$ as a function of the diametral pitch, $P$. Let the values of the diametral pitch correspond to those shown in Figure 5.10 of your text, starting at $P = 8 \, \text{teeth/in}$ and going up to $P = 80 \, \text{teeth/in}$. Let all four plots (one for each gear ratio) appear on the same set of axes. This can be implemented fairly easily in MATLAB.

Repeat with a pressure angle of $\phi = 20^\circ$. A good way to display this second set of plots is side-by-side with the first set making use of MATLAB’s subplot command and setting the range of the vertical axes to be the same, say, 1.5 to 2, in both subplots. This allows for an intuitive comparison of the effect of pressure angle.

As communicators, we should strive to find the clearest ways to show our insights visually.

Summary

For the analytically inclined, it is possible to find the limit of $CR$ as $P$ goes to infinity. The result is

$$CR_{\text{max}} = \frac{4}{\pi} \sin 2\phi$$  \hfill (8)

The proof of this result is left as an optional exercise for the reader. It is interesting that this limit does not depend on the center-to-center distance or the gear ratio.

Some trends that emerge from studying your exercise:

1. For a given pressure angle, $\phi$, there is an upper limit to the maximum achievable contact ratio (See equation (8)).
2. The smaller the pressure angle, $\phi$, the higher the contact ratio.
3. The smaller the teeth, i.e., the larger $P$, the higher the contact ratio.
4. There is a law of diminishing returns with regard to smaller teeth. In other words, as the teeth become smaller and smaller, CR increases, but the amount of increase diminishes.
5. The closer the gear ratio is to unity, the higher the contact ratio.

References

Figure 15.7: From Juvinall and Marshek. Shows relationships among base and pitch circle radii and pressure angle, as seen also in equation (15.11).

Figure 15.4: From Juvinall and Marshek. Shows the base pitch, $p_b$, the “length” of a single tooth, measured along the base circle.
Figure 15.8: From Juvinall and Marshek. A very busy figure. In our own figures, we will extract relevant details for our particular attention. Note that at this instant there are two pairs of teeth in contact; the pinion tooth at \( a \) is just beginning to push along the pressure line on its gear tooth and that at \( c \) is just finishing pushing on its gear tooth. In a further moment, there will be only one pair, as the contact at \( c \) is about to be lost. The numerical value of the contact ratio for this gearset will, therefore, be somewhere between 1 and 2.
Figure 1: A detail extracted from Figure 15.8, showing addenda and base circles and the pressure line. Key intersections are labeled. The pinion rotates counterclockwise driving the gear clockwise. Contact between engaged teeth occurs only on the pressure line between points $a$ and $c$. 
Figure 2: The purple line represents the imaginary inextensible cord that could drive the pinion and gear. The length in green that straddles the pitch point is the path of contact, $L_{ac}$. The other length in green is an arclength of exactly one tooth measured along the base circle, $p_b$. It is the ratio of these two lengths that is the contact ratio, $CR$. 
Figure 3: The law of cosines can be used with reference to the purple triangle to compute the length of the short side, which is $L_{ap}$. The two longer sides are of known lengths, the addendum radius and the pitch radius of the gear. The included obtuse angle is $\frac{\pi}{2} + \phi$, as should be clear from the figure. The green triangle can be used to find $L_{pc}$. 