# Control of Nonlinear Systems with Time-Varying Output Constraints

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# Control of Nonlinear Systems with Time-Varying Output Constraints

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*Abstract*— **This paper presents output tracking control for strict feedback nonlinear systems with time-varying output constraint. A Barrier Lyapunov Function (BLF), which depends explicitly on time, is employed at the outset to prevent the output from violating the time-varying constraint. Specifically, we allow the barrier limit to vary with the desired trajectory in time. Through a change of coordinates for the tracking error, we then eliminate the time dependence, therefore simplifying the analysis. We show that asymptotic output tracking is achieved without violation of the time-varying constraint, and that all closed loop signals remain bounded. The performance of the proposed control is illustrated through a simulation example.**

## I. Introduction

Physical systems are commonly subject to constraints in various forms, such as physical stoppages, saturation, or even performance and safety specifications. If the constraints are violated during operation, then performance degradation, hazards or system damage may occur. Driven by practical needs and theoretical challenges, the rigorous handling of constraints in control design has become an important research topic in recent decades.

To handle constraints in linear systems, many techniques have been developed, most of which are based on notions of set invariance using Lyapunov analysis [1], [2], [3]. Alternatively, Model Predictive Control considers the problem within an optimization framework inherently suitable for handling constraints, by solving an on-line finite horizon open-loop optimal control problem, subject to the system dynamics and constraints (see e.g. [4], [5]). Reference governors, which modulate the reference signal to avoid any violation of system constraints, have also been employed [6], [7].

More recently, Barrier Lyapunov Functions (BLF) have been proposed to deal with constraints, motivated by the approach of tailoring the Lyapunov function according to the requirements of the problem (e.g. [8], [9]). A BLF is different from traditional Lyapunov functions (e.g. quadratic ones) in that it is not radially unbounded, but grows to infinity whenever its arguments approaches some limits. By keeping the BLF bounded in the closed loop system, it is thus guaranteed that the barriers are not transgressed. BLF-based control design has been proposed for nonlinear systems in Brunovsky form [10] and strict feedback form [11], as well as for electrostatic microactuators [12] and electromagnetic oscillators [13]. For these works, the constraints considered are static and do not change over time.

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In this paper, we tackle the problem of time-varying output constraint by employing a BLF that depends explicitly on time. Instead of defining a barrier limit in such a way that it depends on a worst case constant bound of the desired trajectory over time [11], we allow the barrier limit to vary with the desired trajectory in time, making the design less conservative. Then, through a change of coordinates for the tracking error, we eliminate the explicit time dependence, thus simplifying the analysis to one that is similar to that for the static constraint problem considered in [11]. The proposed design is applicable to the static output constraint problem as a special case, where it improves on the results of [11] by enlarging the set of feasible initial outputs.

The remainder of this paper is organized as follows. In Section II, we formulate the problem of tracking control for nonlinear strict feedback systems with time-varying output constraint, and provide an exposition of the main ideas underlying the use of BLFs for constraint satisfaction. Following that, in Section III, we present the control design to ensure that the output of the plant is constrained. The simulation study in Section IV illustrates the performance of the control, and Section V presents concluding remarks.

## II. Problem Formulation and Preliminaries

Throughout this paper, we denote by $\mathbb{R}_+$ the set of nonnegative real numbers, $\| \bullet \|$ the Euclidean vector norm in $\mathbb{R}^m$, and $\lambda_{\max}(\bullet)$ and $\lambda_{\min}(\bullet)$ the maximum and minimum eigenvalues of $\bullet$, respectively. We also denote $\bar{x}_i = [x_1, x_2, ..., x_i]^T$, $\bar{z}_i = [z_1, z_2, ..., z_i]^T$, $z_{i:j} = [z_i, z_{i+1}, ..., z_j]^T$ and $\bar{y}_{d_i} = [y_d^{(1)}, y_d^{(2)}, ..., y_d^{(i)}]^T$, for positive integers $i, j$.

Consider the strict feedback nonlinear system:

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad i = 1, 2, ..., n-1 \\ \dot{x}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n)u \\ y &= x_1 \end{aligned} \tag{1}$$

where $f_1, ..., f_n, g_1, ..., g_n$ are smooth functions, $x_1, ..., x_n$ are the states, $u$ and $y$ are the input and output respectively. The output $y(t)$ is required to satisfy $|y(t)| \leq k_{c_1}(t)\ \forall t \geq 0$, where $k_{c_1}(t)$ is a positive-valued time-varying constraint.

*Assumption 1:* There exist positive constants $K_{c_i}$, $i = 0, 1, ..., n$, such that the time-varying constraint $k_{c_1}(t)$ and its time derivatives satisfy $0 < k_{c_1}(t) \leq K_{c_0}$ and $|k_{c_1}^{(i)}(t)| \leq K_{c_i}$, $i = 1, ..., n$, $\forall t \geq 0$.

*Assumption 2:* There exists a function $Y_0 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $Y_0(t) < k_{c_1}(t)\ \forall t \geq 0$, and positive constants $Y_i$,

$i = 1, ..., n$, such that the desired trajectory $y_d(t)$ and its time derivatives satisfy $|y_d(t)| \leq Y_0(t)$ and $|y_d^{(i)}(t)| \leq Y_i$, $i = 1, ..., n$, $\forall t \geq 0$.

*Assumption 3:* The functions $g_i(\bar{x}_i)$, $i = 1, 2, ..., n$, are known, and there exists a positive constant $g_0$ such that $0 < g_0 \leq |g_i(\bar{x}_i)|$ for $|x_1| < K_{c_0}$. Without loss of generality, we further assume that $g_i(\bar{x}_i)$, $i = 1, ..., n$, are positive for $|x_1| < K_{c_0}$.

The control objective is to track a desired trajectory $y_d(t)$ while ensuring that all closed loop signals are bounded and that the *output constraint is not violated*. To prevent the output from violating the constraint, we employ a Barrier Lyapunov Function, defined as follows.

*Definition 1:* [11] A Barrier Lyapunov Function is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region $\mathcal{D}$ containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $\mathcal{D}$, has the property $V(x) \to \infty$ as $x$ approaches the boundary of $\mathcal{D}$, and satisfies $V(x(t)) \leq b$ $\forall t \geq 0$ along the solution of $\dot{x} = f(x)$ for $x(0) \in \mathcal{D}$ and some positive constant $b$.

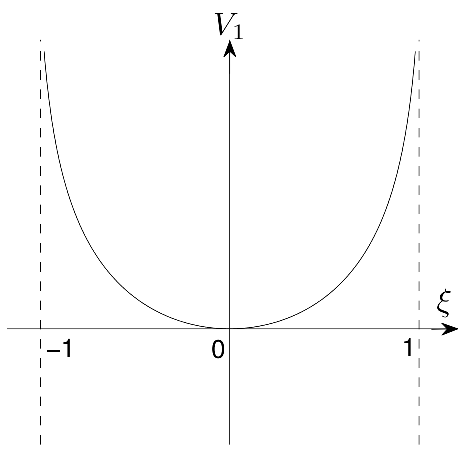


Fig. 1. Schematic illustration of a barrier function $V_1$, which grows to infinity when $|\xi| \to 1$.

A schematic illustration of a Barrier Lyapunov Function is provided in Figure 1. The following lemma formalizes the result for general forms of barrier functions in Lyapunov synthesis satisfying $V_1(\xi) \to \infty$ as $|\xi| \to 1$, and is used to ensure that the time-varying output constraint is not violated.

*Lemma 1:* Let $\mathcal{Z} := \{\xi \in \mathbb{R} : |\xi| < 1\} \subset \mathbb{R}$ and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z} \subset \mathbb{R}^{l+1}$ be open sets. Consider the system

$$\dot{\eta} = h(t, \eta) \tag{2}$$

where $\eta := [w, \xi]^T \in \mathcal{N}$, and $h : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{l+1}$ is piecewise continuous in $t$ and locally Lipschitz in $\eta$, uniformly in $t$, on $\mathbb{R}_+ \times \mathcal{N}$. Suppose that there exist functions $U : \mathbb{R}^l \to \mathbb{R}_+$ and $V_1 : \mathcal{Z} \to \mathbb{R}_+$, continuously differentiable and positive definite in their respective domains, such that

$$V_1(\xi) \to \infty \quad \text{as} \quad |\xi| \to 1 \tag{3}$$

$$\gamma_1(\|w\|) \leq U(w) \leq \gamma_2(\|w\|) \tag{4}$$

where $\gamma_1$ and $\gamma_2$ are class $K_\infty$ functions. Let $V(\eta) := V_1(\xi) + U(w)$, and $\xi(0) \in \mathcal{Z}$. If the inequality holds:

$$\dot{V} = \frac{\partial V}{\partial \eta} h \leq 0 \tag{5}$$

in the set $\xi \in \mathcal{Z}$, then $\xi(t) \in \mathcal{Z}$ $\forall t \in [0, \infty)$.

**Proof:** The proof is similar to that of [11, Lemma 1], and is provided here for completeness.

The conditions on $h$ ensure the existence and uniqueness of a maximal solution $\eta(t)$ on the time interval $[0, \tau_{\max})$, according to [14, p.476 Theorem 54]. From the fact that $\xi(0) \in \mathcal{Z}$, we know that $V_1(\xi(0))$, and thus $V(\eta(0))$, exist.

Since $V(\eta)$ is positive definite and $\dot{V} \leq 0$ in the set $\xi \in \mathcal{Z}$, it follows that $V(\eta(t)) \leq V(\eta(0))$ $\forall t \in [0, \tau_{\max})$. From $V(\eta) = V_1(\xi) + U(w)$ and the fact that $V_1(\xi)$ is a positive function, it is clear that $V_1(\xi(t))$ is bounded $\forall t \in [0, \tau_{\max})$. Since $V_1(\xi) \to \infty$ only if $\xi \to \pm 1$, we conclude, from the boundedness of $V_1(\xi(t))$, that $|\xi(t)| < 1$ $\forall t \in [0, \tau_{\max})$.

Therefore, there is a compact subset $K \subseteq \mathcal{N}$ such that the maximal solution of (2) satisfies $\eta(t) \in K$ $\forall t \in [0, \tau_{\max})$. As a direct consequence of [14, p.481 Proposition C.3.6], we have that $\eta(t)$ is defined $\forall t \in [0, \infty)$. It follows that $\xi(t) \in \mathcal{Z}$ $\forall t \in [0, \infty)$. ■

*Remark 1:* In Lemma 1, we split the state space into $\xi$ and $w$, where $\xi$ is the state to be constrained, and $w$ the free states. The constrained state $\xi$ requires the barrier function $V_1$ to prevent it from reaching the limits $\pm 1$, while the free states may involve quadratic functions.

To establish asymptotic convergence of the signals to zero, we analyze the continuity properties of the derivative of the Lyapunov function candidate in the closed loop and use the following lemma.

*Lemma 2:* [8] (Barbalat's Lemma)
Consider a differentiable function $h(t)$. If $\lim_{t \to \infty} h(t)$ is finite and $\dot{h}$ is uniformly continuous, then $\lim_{t \to \infty} \dot{h}(t) = 0$.

## III. Control Design

The control design is based on backstepping with a barrier function in the first step, followed by quadratic functions in the remaining steps. Unlike [11], where the constraint $k_{c_1}$ is constant, we consider in this paper that $k_{c_1}$ varies with time.

**Step 1:**
Let $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is a stabilizing function. Consider the following barrier function which has an explicit dependence on time:

$$V_1 = \frac{1}{2} \log \frac{k_{b_1}^2(t)}{k_{b_1}^2(t) - z_1^2(t)} \tag{6}$$

where

$$k_{b_1}(t) := k_{c_1}(t) - \bar{y}_d(t) \tag{7}$$

The signal $\bar{y}_d(t)$ is a bound of the desired trajectory $y_d(t)$, and is designed to satisfy the following conditions $\forall t > 0$:

$$\begin{aligned} |y_d(t)| \leq \quad & \bar{y}_d(t) \quad < k_{c_1}(t) \\ & |\bar{y}_d^{(i)}(t)| \quad \leq W_i, \quad i = 1, ..., n \end{aligned} \tag{8}$$

where $W_i$ are positive constants. As a result, we have

$$0 < \underline{k}_{b_1} \le k_{b_1}(t) \le \overline{k}_{b_1}, \quad \forall\, t \ge 0 \tag{9}$$

where $\underline{k}_{b_1}$ and $\overline{k}_{b_1}$ are positive constants defined as

$$\underline{k}_{b_1} := \inf_{t>0}\{k_{c_1}(t) - \bar{y}_d(t)\} \tag{10}$$

$$\overline{k}_{b_1} := \sup_{t>0}\{k_{c_1}(t) - \bar{y}_d(t)\} \tag{11}$$

A possible choice of $\bar{y}_d(t)$, as a function of $y_d(t)$, is

$$\bar{y}_d(t) = \begin{cases} -(2\lambda/\pi)\cos(\pi y_d(t)/2\lambda) + \lambda, & |y_d(t)| \le \lambda \\ |y_d(t)|, & |y_d(t)| > \lambda \end{cases} \tag{12}$$

for any positive constant $\lambda$ satisfying $\lambda < \sup_{t\ge 0} y_d(t)$. This function satisfies the conditions described in (8).

Figure 2 is an illustration of $\bar{y}_d(t)$ described in (12), where, clearly, $|y_d(t)| \le \bar{y}_d(t)\ \forall t > 0$. It is smooth with respect to $y_d$ for all $y_d \in (-\infty, \infty)$, and since $|y_d^{(i)}(t)| < Y_i$ ($i = 1, ..., n$) from Assumption 2, it follows that there exist positive constants $W_i$ such that $|\bar{y}_d^{(i)}(t)| < W_i$ ($i = 1, ..., n$) $\forall\, t > 0$.

This condition is needed because subsequent design steps require its time derivatives to be computed. Note that, if, instead of (7), we had defined $k_{b_1}(t) := k_{c_1}(t) - |y_d(t)|$, then we will face a problem in subsequent design steps since $|y_d|$ is non-differentiable at $y_d = 0$.

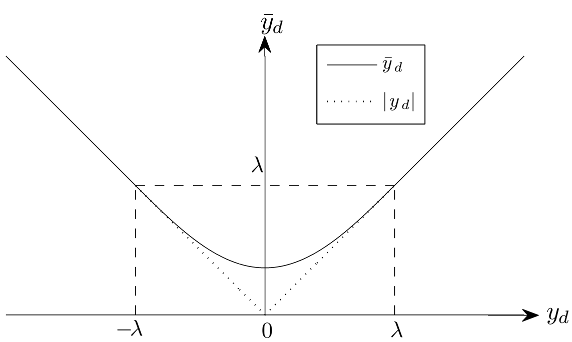

Fig. 2. The function $\bar{y}_d$ is smooth in a $\lambda-$neighborhood of $y_d = 0$.

Now, to remove the explicit dependence on time in (6), we apply a change of coordinates:

$$\xi = \frac{z_1(t)}{k_{b_1}(t)} \tag{13}$$

Hence, we can rewrite $V_1$ in (6) as

$$V_1 = \log \frac{1}{1-\xi} \tag{14}$$

which does not depend explicitly on time, and so satisfies the conditions of Lemma 1, to be used later. It can be shown that

$$\dot{V}_1 = \frac{\xi}{k_{b_1}(1-\xi^2)} \left( f_1 + g_1(z_2 + \alpha_1) - \dot{y}_d - \xi \dot{k}_{b_1} \right)$$

By designing the stabilizing function $\alpha_1$ as

$$\alpha_1 = \frac{1}{g_1} \left( -f_1 - (k_{b_1}\kappa_1 - \dot{k}_{b_1})\xi + \dot{y}_d \right) \tag{15}$$

we obtain

$$\dot{V}_1 = -\frac{\kappa_1 \xi^2}{1-\xi^2} + \frac{g_1 \xi z_2}{k_{b_1}(t)(1-\xi^2)} \tag{16}$$

The first term on the right hand side is non positive in the set $|\xi| < 1$, and the second term is canceled in the second step of the design.

*Remark 2:* From (13), it is observed that the auxiliary state $\xi$ has a singularity at $k_{b_1} = 0$. However, as described in (9), $k_{b_1}(t)$ is bounded away from 0 for all $t > 0$. Therefore, $\xi(t)$ is free from the singularity problem for all $t > 0$.

**Step i** ($i = 2, ..., n$):

Let $z_i = x_i - \alpha_{i-1}$, where $\alpha_{i-1}$ is a stabilizing function. Consider the Lyapunov function candidate

$$V = \sum_{i=1}^{n} V_i \tag{17}$$

where $V_1$ is the barrier function defined in (14), and $V_i = \frac{1}{2} z_i^2$ for $i = 2, ..., n$. Design the stabilizing functions and control law as

$$\alpha_2 = \frac{1}{g_2} \left( -f_2 - \kappa_2 z_2 - \dot{\alpha}_1 - \frac{g_1 \xi}{k_{b_1}(1-\xi^2)} \right) \tag{18}$$

$$\alpha_i = \frac{1}{g_i}(-f_i - \kappa_i z_i - \dot{\alpha}_{i-1} - g_{i-1} z_{i-1}) \tag{19}$$

$$u = \alpha_n \tag{20}$$

where

$$\dot{\alpha}_i = \frac{\partial \alpha_i}{\partial t} + \sum_{j=1}^{i} \frac{\partial \alpha_i}{\partial x_j}(f_j + g_j x_{j+1}) + \sum_{j=0}^{i} \frac{\partial \alpha_i}{\partial y_d^{(j)}} y_d^{(j+1)}, \quad i = 1, ..., n-1 \tag{21}$$

This yields the closed loop system:

$$\dot{\xi} = -\kappa_1 \xi + \frac{g_1 z_2}{k_{b_1}(t)} \tag{22}$$

$$\dot{z}_2 = -\kappa_2 z_2 - \frac{g_1 \xi}{k_{b_1}(t)(1-\xi^2)} + g_2 z_3 \tag{23}$$

$$\dot{z}_i = -\kappa_i z_i - g_i z_{i+1}, \quad i = 3, ..., n \tag{24}$$

and

$$\dot{V} = -\frac{\kappa_1 \xi^2}{1-\xi^2} - \sum_{j=2}^{n} \kappa_j z_j^2 \tag{25}$$

Let the closed loop system (22)-(24) be written as

$$\dot{\eta} = h(t, \eta) \tag{26}$$

where $\eta = [\xi, z_2, ..., z_n]^T$. Since $k_{b_1}(t) > 0\ \forall t > 0$, $h(t, \eta)$ is piecewise continuous in $t$ and locally Lipschitz in $\eta$, uniformly in $t$, on $\mathbb{R}_+ \times \mathcal{N}$, where

$$\mathcal{N} := \{\xi, z_2, ..., z_n \in \mathbb{R}^n : |\xi| < 1\} \tag{27}$$

Then, together with the fact that $\dot{V} \le 0$ in the set $\xi \in \mathcal{Z}$, and that $\xi(0) \in \mathcal{Z}$, where $\mathcal{Z} := \{\xi \in \mathbb{R} : |\xi| < 1\} \subset \mathbb{R}$, we invoke Lemma 1 to obtain that $\xi(t) \in \mathcal{Z}$ for all $t > 0$. Since $k_{b_1}(t) > 0 \ \forall \ t > 0$, we infer, from (13), that

$$|z_1(t)| < k_{b_1}(t), \quad \forall \ t > 0 \tag{28}$$

*Remark 3:* From (18)-(19), there appears to be a possibility of $\alpha_i$ becoming unbounded if $|\xi(t)| = 1$ at some $t$. However, this will not happen since we have shown that $|\xi(t)| < 1$. In fact, it is further shown in Theorem 1 that $|\xi(t)| \leq C$, where $C$ is a positive constant that is dependent on the initial conditions. Thus, the stabilizing functions $\alpha_2(t), ..., \alpha_{n-1}(t)$ and control $u(t)$ remain bounded for all $t > 0$.

We are now ready to show the main results.

*Theorem 1:* Consider the closed loop system (1), (15) and (18)-(20) under Assumptions 1-3. If the initial output $y(0)$ satisfies

$$|y(0) - y_d(0)| < k_{b_1}(0) \tag{29}$$

then the following properties hold.

i) The signals $\xi(t)$ and $z_i(t)$, $i = 1, 2, ..., n$, are bounded, for all $t > 0$, as follows:

$$|\xi(t)| \leq \sqrt{1 - e^{-2V(0)}} \tag{30}$$

$$|z_1(t)| \leq D_{z_1}(t) \leq \overline{k}_{b_1} \tag{31}$$

$$\|z_{2:n}(t)\| \leq \sqrt{2V(0)} \tag{32}$$

where is $\overline{k}_{b_1}$ defined in (11), and

$$D_{z_1}(t) := k_{b_1}(t)\sqrt{1 - e^{-2V(0)}} \tag{33}$$

ii) The output $y(t)$ is bounded as follows:

$$|y(t)| \leq D_{z_1}(t) + A_0(t) < k_{c_1}(t), \quad \forall t \geq 0 \tag{34}$$

i.e. the output constraint is never violated.

iii) All closed loop signals are bounded.

iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.

**Proof:**

i) Lemma 1 yields $|\xi(t)| < 1$, from which we know that $|z_1(t)| < k_{b_1}(t) \ \forall \ t > 0$. Then, from (25), we have $V(t) \leq V(0) \ \forall \ t > 0$, which implies that

$$\frac{1}{2} \log \frac{k_{b_1}^2(t)}{k_{b_1}^2(t) - z_1^2(t)} \leq V(0) \tag{35}$$

Taking exponentials on both sides of the inequality and rearranging, we obtain

$$k_{b_1}^2(t) \leq e^{2V(0)}(k_{b_1}^2(t) - z_1^2(t)) \tag{36}$$

which leads to

$$|z_1(t)| \leq k_{b_1}(t)\sqrt{1 - e^{-2V(0)}}, \quad \forall \ t > 0$$

Similarly, from the fact that

$$\frac{1}{2} \sum_{j=2}^{n} z_j^2(t) \leq V(0) \tag{37}$$

we can show that $\|z_{2:n}(t)\| \leq \sqrt{2V(0)} \ \forall \ t > 0$.

ii) Since $|z_1(t)| \leq D_{z_1}(t) < k_{b_1}(t)$, and $|y_d(t)| \leq A_0(t)$, we infer, from $y(t) = z_1(t) + y_d(t)$, that

$$\begin{aligned} |y(t)| &\leq |z_1(t)| + |y_d(t)| \\ &\leq D_{z_1}(t) + \bar{y}_d(t) \\ &< k_{b_1}(t) + \bar{y}_d(t) = k_{c_1}(t) \end{aligned} \tag{38}$$

Hence, we conclude that the time-varying output constraint is never violated.

iii) From (i), we know that the error signals $z_1(t)$, ..., $z_n(t)$ are bounded. Boundedness of $z_1(t)$ and $y_d(t)$ implies that the state $x_1(t)$ is bounded.
From (7), we know that $\dot{k}_{b_1}(t)$ is bounded, since $\dot{k}_{c_1}(t) \leq K_{c_1}$ from Assumption 1, and $\dot{\bar{y}}_d(t) \leq W_1$ from (8), where $K_{c_1}$ and $W_1$ are some positive constants. Together with the fact that $\dot{y}_d(t)$ is bounded from Assumption 2, it is clear, from (15), that the stabilizing function $\alpha_1(t)$ is also bounded. This leads to boundedness of $x_2(t)$, from $x_2 = z_2 + \alpha_1$.
Since $|\xi(t)| < 1 \ \forall \ t > 0$, and $\alpha_2$ is a continuous function of the bounded signals $\bar{x}_2(t)$, $\bar{z}_2(t)$, and $\bar{y}_{d_2}(t)$ in the set $|\xi| < 1$, we know that $\alpha_2(t)$ is bounded. This, in turn, leads to boundedness of state $x_3(t)$, since $x_3 = z_3 + \alpha_2$.
Following this line of argument, we can progressively show that each $\alpha_i(t)$, for $i = 3, ..., n-1$, is bounded, since it is a continuous function of the bounded signals $\bar{x}_i(t), \bar{z}_i(t)$, and $\bar{y}_{d_i}(t)$ in the set $z_1 \in (-k_{b_1}, k_{b_1})$. Thus, the boundedness of state $x_{i+1}(t)$ can be shown. With $\bar{x}_n(t)$, $\bar{z}_n(t)$ bounded, and $|\xi(t)| < 1 \ \forall \ t > 0$, we conclude that the control $u(t)$ is bounded. Hence, all closed loop signals are bounded.

iv) Let $\rho := \kappa_1 \xi^2/(1 - \xi^2)$, which is differentiable in the set $|\xi| < 1$. Since $|\xi(t)| < 1 \ \forall \ t > 0$ from Lemma 1, we can integrate both sides of (25) to obtain

$$\lim_{t \to \infty} \int_0^t \rho(\tau) \ d\tau \leq V(0) < \infty \tag{39}$$

Next, based on (22) and the bounds $|x_1(t)| \leq K_{c_0}$, $|\xi(t)| < 1$, $z_2(t) \leq \sqrt{2V(0)}$, and $k_{b_1}(t) \geq \underline{k}_{b_1}$, it can be shown that $\dot{\rho}(t)$ is bounded, thus implying that $\rho(t)$ is uniformly continuous.
Then, by Lemma 2, we obtain that $\rho(t) \to 0$ as $t \to \infty$, and thus, $\xi(t) \to 0$ as $t \to \infty$. Since $\xi(t) = z_1(t)/k_{b_1}(t)$ and $k_{b_1}(t) \geq \underline{k}_{b_1} > 0 \ \forall \ t > 0$, we finally have $z_1(t) \to 0$ as $t \to \infty$. ■

*Remark 4:* The BLF-based control proposed in this paper can be extended to deal with uncertainty in linearly parameterizable nonlinearities $f_i(\bar{x}_i) = \theta^T \psi(\bar{x}_i)$, $i = 1, .., n$, using adaptive backstepping control techniques, along the lines of [11].

*Remark 5:* Although we focused on a symmetric barrier function in this paper, the design can be extended for an asymmetric barrier function:

$$V_1 = q(z_1) \log \frac{1}{1 - \xi_a} + (1 - q(z_1)) \log \frac{1}{1 - \xi_b} \tag{40}$$

where $\xi_a := z_1/k_{a_1}(t)$, $\xi_b := z_1/k_{b_1}(t)$, the barrier limits $k_{a_1}(t), k_{b_1}(t)$ not necessarily equal for any $t \geq 0$, and

$$q(\bullet) \quad := \quad \begin{cases} 1, & \text{if } \bullet > 0 \\ 0, & \text{if } \bullet \leq 0 \end{cases} \tag{41}$$

Thereafter, we can combine the approach in this paper with that of [11]. A nice property of the asymmetry is that the set of feasible initial outputs can be enlarged to $-k_{a_1}(0) < z_1(0) < k_{b_1}(0)$ instead of $|z_1(0)| < \min\{k_{a_1}(0), k_{b_1}(0)\}$.

*Remark 6:* For the special case of static output constraint, where $k_{c_1}$ is constant, the proposed control is also applicable. It improves on the results of [11] by enlarging the set of feasible initial output from

$$|y(0) - y_d(0)| < k_{c_1} - \max_{t\geq 0}\{|y_d(t)|\} \tag{42}$$

which is based on a worst case bound of $y_d(t)\ \forall\ t \geq 0$, to

$$|y(0) - y_d(0)| < k_{c_1} - \bar{y}_d(0) \tag{43}$$

where $\bar{y}_d(0) \leq \max_{t\geq 0}\{|y_d(t)|$.

## IV. Simulation

In this section, we present simulation studies to illustrate the performance of the proposed control. Consider the following second-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= 0.1x_1^2 + x_2 \\ \dot{x}_2 &= 0.1x_1x_2 - 0.2x_1 + (1 + x_1^2)u \\ y &= x_1 \end{aligned} \tag{44}$$

The objective is for $y(t)$ to track a desired trajectory

$$y_d(t) = 0.5 \sin t \tag{45}$$

subject to the output constraint

$$|y(t)| < k_{c_1}(t) = 1 + 0.1 \cos t, \quad \forall\ t > 0 \tag{46}$$

The design parameters are selected as $\kappa_1 = \kappa_2 = 2$ and $\lambda = 0.1$. First, we construct $\bar{y}_d(t)$ according to (12), and then compute $k_{b_1}(t)$ based on knowledge of $k_{c_1}(t)$ and $\bar{y}_d(t)$, as

$$k_{b_1}(t) = \begin{cases} 1 + 0.1\cos t + \frac{2\lambda}{\pi}\cos(\frac{\pi y_d(t)}{2\lambda}) - \lambda, & |y_d(t)| \leq \lambda \\ 1 + 0.1\cos t - |y_d(t)|, & |y_d(t)| > \lambda \end{cases} \tag{47}$$

Figure 3 shows that $\bar{y}_d(t)$ is a smoothed version of $|y_d(t)|$ within a neighborhood of 0. Additionally, we have $\bar{y}_d(t) \geq y_d(t)$, which is important for ensuring that $|y(t)| < k_{c_1}$, as follows from (38). The smoothness allows $k_{b_1}(t) = k_{c_1}(t) - \bar{y}_d(t)$ to be differentiated so that the stabilizing functions and control law are well-defined. Furthermore, the time derivatives of $\bar{y}_d(t)$ are bounded, as shown in Figure 4.

We simulate the system response under the proposed BLF-based control

$$\begin{aligned} \alpha_1 &= -0.1x_1^2 - (2k_{b_1} - \dot{k}_{b_1})\xi + 0.5\cos t \\ u &= \frac{1}{1 + x_1^2}\left(-\frac{\xi}{k_{b_1}(1 - \xi^2)} - 0.1x_1x_2 + 0.2x_1 \right. \\ &\quad \left. -2z_2 - \dot{\alpha}_1\right) \end{aligned} \tag{48}$$

where $\xi = z_1/k_{b_1}$. According to (29), with $y_d(0) = 0$ and $\lambda = 0.1$, the initial output is required to satisfy

$$|y(0)| < 1.1 + \frac{2\lambda}{\pi} - \lambda = 1.0637 \tag{49}$$

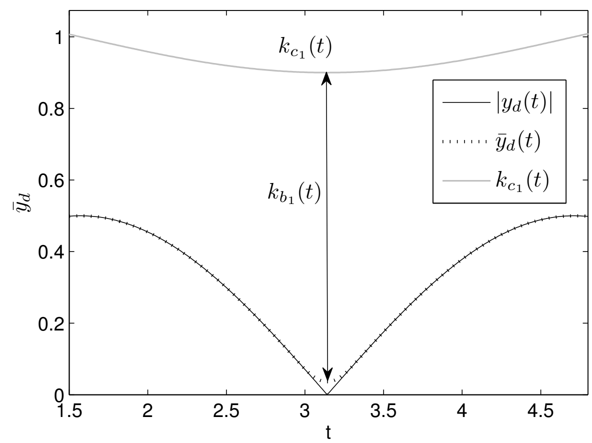


Fig. 3. The signal $\bar{y}_d(t)$ is a smoothed version of $|y_d(t)|$, such that $k_{b_1}(t) = k_{c_1}(t) - \bar{y}_d(t)$ is differentiable.

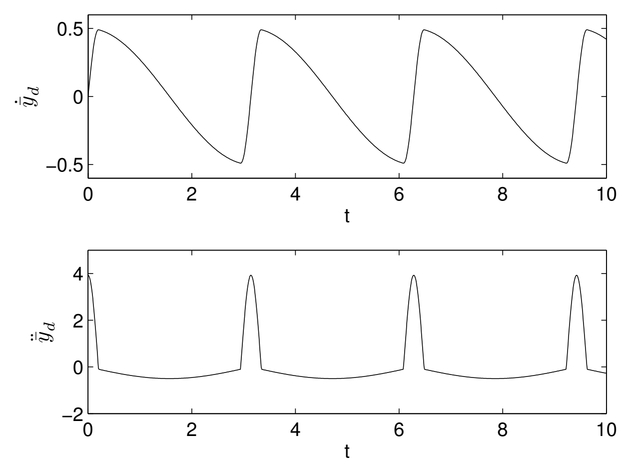


Fig. 4. The first and second order time derivatives of the signal $\bar{y}_d(t)$ are bounded.

To the illustrate the effect of the BLF-based control, we consider two representative initial points, $x(0) = (0.5, 4.5)$ and $x(0) = (-0.5, -4)$, which yield a large transient tracking error such that $y(t)$ reaches a proximity of $k_{c_1}(t)$ initially. Despite the tendency to transgress the constraint, Figure 5 shows that the output trajectories never violate the constraints, i.e. they satisfy $|y(t)| < k_{c_1}(t)$ for all $t > 0$. Furthermore, they converge to the desired trajectory $y_d(t)$.

As shown in Figure 6, the trajectories of the tracking error $z_1(t)$ converge to 0 while satisfying $|z_1(t)| < k_{b_1}(t)$ for all $t > 0$, despite the fact that they come very close to $k_{b_1}(t)$ initially. That $|z_1(t)| < k_{b_1}(t)$ is key to ensuring $|y(t)| < k_{c_1}(t)$, as shown in (38).

The control signals $u(t)$ are shown in Figure 7. Peaking effect of $u(t)$ is observed initially, as a result of the auxiliary state $\xi(t)$ coming close to the limits at $\xi = \pm 1$. As seen in (48), $u(t)$ can grow large when $|\xi(t)| \to 1$. As $\xi(t)$ moves away from $\xi = \pm 1$, the control signal drops in magnitude significantly.

## V. Conclusions

In this paper, we have presented a control for strict feedback nonlinear systems with time-varying output constraints to achieve output tracking. We have employed a Barrier Lyapunov Function, which grows to infinity whenever its arguments approaches some limits, to prevent transgression of the output constraint. We have shown that asymptotic output tracking is achieved, that the output remains within the constrained region, and that all closed loop signals are bounded. The performance of the proposed control has been illustrated through a simulation example.

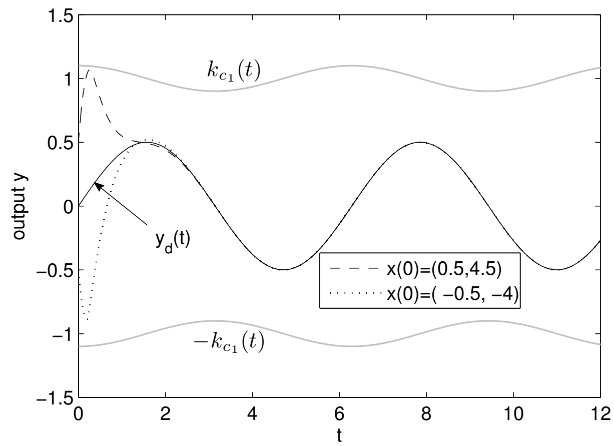


Fig. 5. The output trajectories $y(t)$, corresponding to two representative initial points $x(0) = (0.5, 4.5)$ and $x(0) = (-0.5, -4)$, converge to the desired trajectory $y_d(t)$ while satisfying $y(t) < k_{c_1}(t)$ for all $t > 0$.

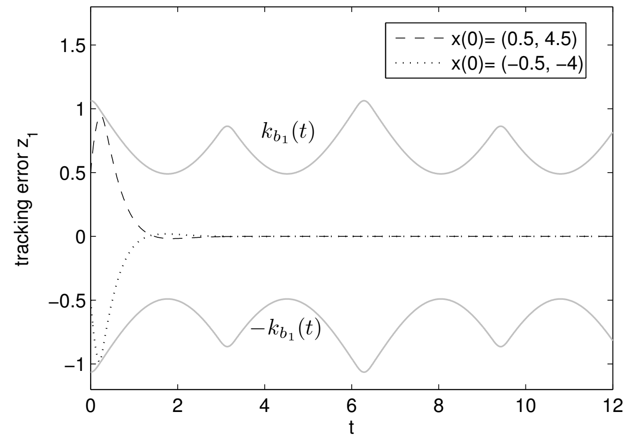


Fig. 6. The tracking error trajectories $z_1(t)$ converge to 0 while satisfying $z_1(t) < k_{b_1}(t)$ for all $t > 0$.

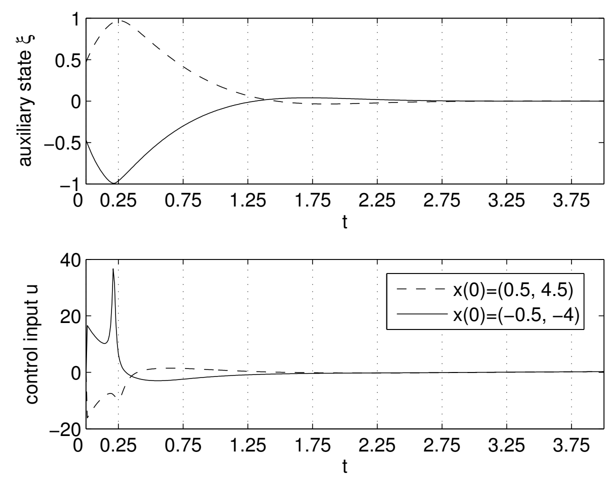


Fig. 7. The auxiliary state $\xi(t)$ is constrained in the set $|\xi| < 1$, and the control input $u$ remains bounded.

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