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Stabilization of an ODE-Schrödinger Cascade

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Abstract— **In this paper, we consider a problem of stabilization of an ODE-Schrödinger cascade, where the interconnection between them is bi-directional at a single point. By using the backstepping approach, which uses an invertible Volterra integral transformation together with the boundary feedback to convert the unstable plant into a well-damped target system, the target system in our case is given as a PDE-ODE cascade with exponential stability at the pre-designed decay rate. Instead of one-step backstepping control, which results in difficulty in finding the kernels, we develop a two-step backstepping control design by introducing an intermediate target system and an intermediate control. The exponential stability of the closed-loop system is investigated using the Lyapunov method. A numerical simulation is provided to illustrate the effectiveness of the proposed design.**

I. Introduction

Backstepping approach, which was originally developed for parabolic PDEs [1], [2], has recently been applied to control problems for PDE-ODE cascades [3], [4], [5], [6], [7], with applications of interest including fluids, structures, thermal, chemically-reacting, and plasma systems. This method uses an invertible Volterra integral transformation together with the boundary feedback to convert the unstable plant into a well-damped target system. The kernel of this transformation satisfies a certain PDE and ODE, which turn out to be solvable in closed form. For PDE-ODE cascades in [3], [4], [5], [6], [7], the interconnection between the PDE and the ODE is one directional. For example, in [5], the dynamics at the input of the ODE is governed by a heat equation, whereas the ODE does not act on the PDE. In some cases, the interconnection between them could be bi-directional, e.g., the PDE and ODE are coupled with each other. Control of coupled PDE-ODE systems has received attention recently, [8], where state and output feedback boundary control has been designed for a coupled heat PDE and ODE system.

In this paper, we consider a problem of stabilization of an ODE-Schrödinger equation cascade, where the interconnection between them is bi-directional at a single point. Both states of ODE and Schrödinger equation are considered as complex numbers. The motivation for this kind of problem can be provided in the context of various applications in quantum mechanics, chemical process control, and other areas. As in [9], [10], the Schrödinger equation is usually considered as a complex-valued heat equation such that the backstepping method developed for parabolic PDEs could be applied. Motivated by two-step backstepping transformations in [5], which was adopted to improve performance and achieve exponential stability with arbitrarily fast decay rate, we adopt two-step backstepping control design to make the system stable in this paper. Instead of one-step backstepping control, which results in difficulty in finding the kernels, we develop a two-step backstepping control design by introducing an intermediate target system and an intermediate control. First, we design the comprehensive control to convert the original system into the intermediate target system. Secondly, we design the intermediate control to convert the intermediate target system into the final target system.

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The rest of the paper is organized as follows. The problem is formulated in Section II. In Section III, the two-step backstepping control is developed and the stability analysis of the closed-loop system is provided by Lyapunov method. Simulation example is presented in Section IV followed by the concluding remarks in Section V.

II. Problem Formulation

Consider an ODE-Schrödinger equation cascade as in (1)-(4), where the interconnection between them is bi-directional at a single point:

$$\dot{X}(t) = AX(t) + Bu(0,t),\ t > 0, \quad (1)$$
$$u_t(x,t) = -iu_{xx}(x,t),\ x \in (0,1),\ t > 0, \quad (2)$$
$$u_x(0,t) = CX(t), \quad (3)$$
$$u(1,t) = U(t) \quad (4)$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$, $C \in \mathbb{C}^{1\times n}$, $X(t) \in \mathbb{C}^{n\times 1}$ is the state of ordinary differential equation , $u(x,t) \in \mathbb{C}$ is the displacement of Schrödinger equation, and $U(t) \in \mathbb{C}$ is the control force to the entire system. The whole system is depicted in Fig. 1.

The control objective is to find a feedback law $U(t)$, so that the controlled system (1)-(4) is exponentially stable. To achieve this, a new cascaded ODE-PDE target system is introduced in the form:

$$\dot{X}(t) = (A+BK)X(t) + Bz(0,t), \quad (5)$$
$$z_t(x,t) = -iz_{xx}(x,t) - cz(x,t),\ c > 0, \quad (6)$$
$$z_x(0,t) = 0, \quad (7)$$
$$z(1,t) = 0 \quad (8)$$

where $z(x,t) \in \mathbb{C}$, and we assume that the pair (A,B) is stabilizable and take $K \in \mathbb{C}^{1\times n}$ to be a known vector such that $A+BK$ is Hurwitz. It is easy to know that the target system (5)-(8) is exponential stable with arbitrarily fast decay rate.

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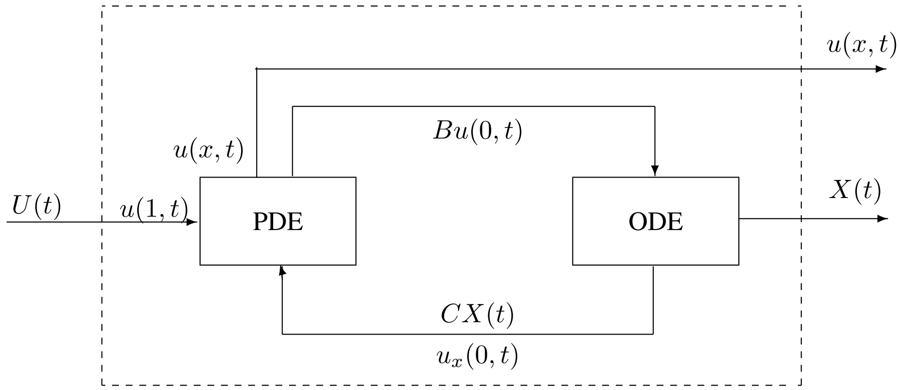

Fig. 1. Block diagram for the coupled ODE-PDE system

III. Backstepping Design

In this section, we seek the boundary controller $U(t)$ in (4) to exponentially stabilize the system (X,u) (1)-(4) into the target system (X,z) (5)-(8) using the backstepping control design for parabolic PDEs [1], [2]. The method uses invertible Volterra integral transformation together with the boundary feedback to convert the unstable plant (X,u) (1)-(4) into a well-damped target system (X,z) (5)-(8). For this we use a change of variables based on a Volterra series. However, the one-step backstepping transformation for $(X,u) \longmapsto (X,z)$ results in difficulty in finding the kernels. To avoid this, we introduce an intermediate target system (X,w):

$$\begin{aligned} \dot{X}(t) &= (A+BK)X(t)+Bw(0,t), && (9)\\ w_t(x,t) &= -iw_{xx}(x,t), && (10)\\ w_x(0,t) &= 0, && (11)\\ w(1,t) &= W(t) && (12) \end{aligned}$$

where $w(x,t) \in \mathbb{C}$, and $W(t) \in \mathbb{C}$ is the intermediate control.

The main idea is using two-step backstepping control design as shown in Fig. 2: (i) design the comprehensive control $U(t)$ to convert the original system into the intermediate target system; and (ii) design the intermediate control $W(t)$ to convert the intermediate target system (X,w) into the final target system (X,z).

A. Design for System (X,u) to (X,w)

In this subsection, we look for the backstepping control $U(t)$ and the transformation $(X,u) \longmapsto (X,w)$ between systems (1)-(4) and (9)-(12). As in [9], [10], the Schrödinger equation is usually considered as a complex-valued heat equation such that the backstepping method developed for parabolic PDEs [1] could be applied.

We postulate the transformation $(X,u) \longmapsto (X,w)$ in the form

$$\begin{aligned} X(t) &= X(t) && (13)\\ w(x,t) &= u(x,t) - \int_0^x q(x,y)u(y,t)dy - \gamma(x)X(t) && (14) \end{aligned}$$

where kernels $q(x,y)$ and $\gamma(x)$ to be derived.

Differentiating (14) once and twice with respect to x, we get, respectively,

$$\begin{aligned} w_x(x,t) &= u_x(x,t) - q(x,x)u(x,t)\\ &\quad - \int_0^x q_x(x,y)u(y,t)dy - \gamma'(x)X(t) && (15)\\ w_{xx}(x,t) &= u_{xx}(x,t) - q(x,x)u_x(x,t)\\ &\quad - \left(q'(x,x)+q_x(x,x)\right)u(x,t)\\ &\quad - \int_0^x q_{xx}(x,y)u(y,t)dy - \gamma''(x)X(t) && (16) \end{aligned}$$

where $q'(x,x) = q_x(x,x) + q_y(x,x)$.

The first derivative of $w(x,t)$ with respect to t is

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x q(x,y)u_t(y,t)dy\\ &\quad -\gamma(x)(AX(t)+Bu(0,t))\\ &= -iu_{xx}(x,t) + iq(x,x)u_x(x,t)\\ &\quad -iq_y(x,x)u(x,t) + i\int_0^x q_{yy}(x,y)u(y,t)dy\\ &\quad + \left(iq_y(x,0) - \gamma(x)B\right)u(0,t)\\ &\quad -(\gamma(x)A + iq(x,0)C)X(t) && (17) \end{aligned}$$

Let us now examine the expressions:

$$\begin{aligned} w(0,t) &= u(0,t) - \gamma(0)X(t) && (18)\\ w_x(0,t) &= (C-\gamma'(0))X(t)\\ &\quad -q(0,0)u(0,t) && (19)\\ w_t(x,t)+iw_{xx}(x,t) &= -2iq'(x,x)u(x,t) - (iq(x,0)C\\ &\quad +\gamma(x)A + i\gamma''(x))X(t) \end{aligned}$$

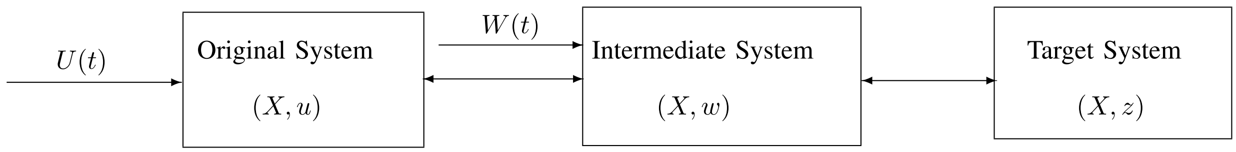

Fig. 2. Block diagram for two-step backstepping control design.

$$
\begin{aligned}
&+(iq_y(x,0) - \gamma(x)B)u(0,t) \\
&+i\int_0^x (q_{yy}(x,y) \\
&-q_{xx}(x,y))u(y,t)dy \qquad (20)
\end{aligned}
$$

A sufficient condition for (9)-(12) to hold for any continuous functions $u(x,t)$ and $X(t)$ is that $\gamma(x)$ and $q(x,y)$ satisfy

$$
\begin{aligned}
\gamma''(x) - i\gamma(x)A + q(x,0)C &= 0, && (21)\\
\gamma(0) &= K, && (22)\\
\gamma'(0) &= C, && (23)
\end{aligned}
$$

which represents a second-order ODE in x, and

$$
\begin{aligned}
q_{xx}(x,y) &= q_{yy}(x,y), && (24)\\
q(x,x) &= 0, && (25)\\
q_y(x,0) &= -i\gamma(x)B && (26)
\end{aligned}
$$

which is a second-order hyperbolic PDE of the Goursat type. We then proceed to solve this coupled system explicitly. The general solution of the PDE (24)-(26) is in the following form [11]

$$q(x,y) = \phi(x-y) + \zeta(x+y) \qquad (27)$$

which, using the boundary conditions, yields

$$
\begin{aligned}
\zeta(2x) &= 0 && (28)\\
-\phi'(x) &= -i\gamma(x)B && (29)
\end{aligned}
$$

Integrating (29), we obtain

$$\phi(x) = \int_0^x i\gamma(\sigma)Bd\sigma$$

Thus, we have

$$q(x,y) = \int_0^{x-y} i\gamma(\sigma)Bd\sigma \qquad (30)$$

Substituting (30) into (21) leads to

$$\gamma''(x) - i\gamma(x)A + \int_0^x i\gamma(\sigma)Bd\sigma C = 0 \qquad (31)$$

Differentiating (31) once, we get the third order ODE

$$\gamma^{(3)}(x) - i\gamma'(x)A + i\gamma(x)BC = 0 \qquad (32)$$

and initial values

$$\gamma''(0) = iKA, \ \gamma'(0) = C, \ \ \gamma(0) = K.$$

Define

$$\Omega(x) = [\gamma(x) \ \gamma'(x) \ \gamma''(x)]$$

and

$$D = \begin{bmatrix} 0 & 0 & -iBC \\ I & 0 & iA \\ 0 & I & 0 \end{bmatrix}$$

then (32) is written into

$$\Omega'(x) = \Omega(x)D$$

Its solution is

$$\Omega(x) = \Omega(0)e^{Dx}$$

with

$$\Omega(0) = [K \ \ C \ \ iKA].$$

Hence, the solution to the ODE (21)-(23) is

$$\gamma(x) = [K \ \ C \ \ iKA]e^{\begin{bmatrix} 0 & 0 & -iBC \\ I & 0 & iA \\ 0 & I & 0 \end{bmatrix}x} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \qquad (33)$$

The transformation $(X,u) \longmapsto (X,w)$ (13)-(14) is invertible, and the inverse transformation $(X,w) \longmapsto (X,u)$ is postulated in the following form

$$
\begin{aligned}
X(t) &= X(t) && (34)\\
u(x,t) &= w(x,t) + \int_0^x \iota(x,y)w(y,t)dy + \psi(x)X(t) && (35)
\end{aligned}
$$

where kernels $\iota(x,y)$ and $\psi(x)$ to be derived.

In a similar manner to finding the kernels $q(x,y)$ and $\gamma(x)$ of the direct transformation, the derivatives u_x, u_{xx} and u_t are computed, and a sufficient condition for (1)-(3) to hold is that $\iota(x,y)$ $(0 \le y \le x \le 1)$ and $\psi(x)$ $(0 \le x \le 1)$ satisfy

$$
\begin{aligned}
\psi''(x) - i\psi(x)(A+BK) &= 0, && (36)\\
\psi(0) &= K, && (37)\\
\psi'(0) &= C, && (38)
\end{aligned}
$$

and

$$
\begin{aligned}
\iota_{xx}(x,y) &= \iota_{yy}(x,y), && (39)\\
\iota(x,x) &= 0, && (40)\\
\iota_y(x,0) &= -i\psi(x)B && (41)
\end{aligned}
$$

This cascade system can be solved explicitly. The solution to the ODE (36)-(38) is

$$\psi(x) = [K \quad C] e^{\begin{bmatrix} 0 & i(A+BK) \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

and the explicit solution to the PDE (39)-(41) is

$$\iota(x,y) = \int_0^{x-y} i\psi(\sigma) B d\sigma \tag{42}$$

Thus, the control law is obtained by setting $x = 1$ in (14):

$$\begin{aligned} U(t) &= u(1,t) \\ &= W(t) + \int_0^1 q(1,y)u(y,t)dy + \gamma(1)X(t) \end{aligned} \tag{43}$$

where $q(x,y)$ and $\gamma(x)$ are defined in (30) and (33) respectively, and the intermediate control $W(t)$ is to be defined in the next subsection.

B. Design for System (X, w) to (X, z)

In this subsection, we look for the backstepping control $W(t)$ and the transformation $(X, w) \longmapsto (X, z)$ between the intermediate target system (9)-(12) and the final target system (5)-(8).

Taking the diffusion coefficient as $-i$ in the reaction-advection-diffusion equation [1] (with the advection and reaction coefficients being zero) and following the control design developed there, we use the transformation $(X, w) \longmapsto (X, z)$ in the form

$$X(t) = X(t) \tag{44}$$

$$z(x,t) = w(x,t) - \int_0^x \kappa(x,y)w(y,t)dy \tag{45}$$

along with the intermediate controller $W(t)$ at $x = 1$:

$$W(t) = w(1,t) = \int_0^1 \kappa(1,y)w(y,t)dy \tag{46}$$

where $\kappa(x,y)$ ia a complex-valued control gain.

The gain kernel PDE takes the form

$$\kappa_{xx}(x,y) - \kappa_{yy}(x,y) = ci\kappa(x,y), \tag{47}$$

$$\kappa_y(x,0) = 0, \tag{48}$$

$$\kappa(x,x) = -\frac{ci}{2}x. \tag{49}$$

The explicit solution to the PDE (47)-(49) is given as [1]:

$$\begin{aligned} \kappa(x,y) &= -cix \frac{I_1(\sqrt{ci(x^2-y^2)})}{\sqrt{ci(x^2-y^2)}} \\ &= x\sqrt{\frac{c}{2(x^2-y^2)}} \Big[(i-1)\mathrm{ber}_1\left(\sqrt{c(x^2-y^2)}\right) \\ &\quad -(i+1)\mathrm{bei}_1\left(\sqrt{c(x^2-y^2)}\right)\Big], \end{aligned} \tag{50}$$

where $I_1(\cdot)$ is the modified Bessel function of order one and $\mathrm{ber}_1(\cdot)$ and $\mathrm{bei}_1(\cdot)$ are the Kelvin functions, which are defined as

$$\mathrm{ber}_1(x) = -\mathrm{Im}\left\{ I_1\left(\frac{1+i}{\sqrt{2}}x\right)\right\},$$

$$\mathrm{bei}_1(x) = \mathrm{Re}\left\{ I_1\left(\frac{1+i}{\sqrt{2}}x\right)\right\}.$$

In a similar manner to finding the kernel $\kappa(x,y)$, the inverse of the transformation (45) can be found as follows

$$w(x,t) = z(x,t) + \int_0^x p(x,y)z(y,t)dy \tag{51}$$

where

$$\begin{aligned} p(x,y) &= -cix \frac{J_1(\sqrt{ci(x^2-y^2)})}{\sqrt{ci(x^2-y^2)}} \\ &= -cix - cix \sum_{m=1}^{\infty} \frac{(-1)^m (ci(x^2-y^2))^m}{4^m m!(m+1)!}, \\ &\qquad 0 \le y \le x \le 1, \end{aligned} \tag{52}$$

and J_1 is a Bessel function.

We are now ready to state our main result:

Theorem 1: Consider the closed-loop system consisting of the plant (1)-(4) and the control law (43) together with (30)(33)(46)(50). Let the pair (A, B) be stabilizable and choose $K \in \mathbb{C}^{1\times n}$ such that $A+BK$ is Hurwitz. For any initial condition $u(\cdot,0) \in H_1(0,1)$, the closed-loop system has a unique solution $(X(t), u(\cdot,t)) \in C([0,\infty], \mathbb{C}^n \times H_1(0,1))$ which is exponentially stable in the sense of the norm

$$\left(|X(t)|^2 + \|u(x,t)\|^2 + \|u_x(x,t)\|^2\right)^{1/2},$$

for all $t \ge 0$, where the symbol $\|\cdot\|$ stands for the $L^2(0,1)$ norm of complex-valued functions.

Proof: Denote

$$\Gamma(s) = \int_0^s \gamma(\sigma) B d\sigma \tag{53}$$

$$\Psi(s) = \int_0^s \psi(\sigma) B d\sigma \tag{54}$$

then the direct and inverse backstepping transformations (14) and (35) can be written as

$$w(x,t) = u(x,t) - \int_0^x \Gamma(x-y)u(y,t)dy - \gamma(x)X(t) \tag{55}$$

$$u(x,t) = w(x,t) + \int_0^x \Psi(x-y)w(y,t)dy + \psi(x)X(t) \tag{56}$$

Taking the L^2-norm for both sides of (55), we have

$$\begin{aligned} \|w\|^2 &= \int_0^1 [u(x) - (\Gamma * u)(x) - \gamma(x)X] \\ &\qquad * \overline{[u(x) - (\Gamma * u)(x) - \gamma(x)X]} dx \\ &\le 3\int_0^1 \left[|u(x)|^2 + |(\Gamma * u)(x)|^2 + |\gamma(x)X|^2\right] dx \\ &\le \alpha_1 \|u\|^2 + \alpha_2 \|X\|^2 \end{aligned} \tag{57}$$

where Cauchy-Schwarz Inequality is used and

$$\alpha_1 = 3(1+\|\Gamma\|^2),\ \alpha_2 = 3\|\gamma\|^2.$$

Similarly, for (56), we have

$$\|u\|^2 \leq \beta_1\|w\|^2 + \beta_2|X|^2 \tag{58}$$

where

$$\beta_1 = 3(1+\|\Psi\|^2),\ \beta_2 = 3\|\psi\|^2.$$

Also, from

$$w_x(x,t) = u_x(x,t) - \int_0^x \Gamma_x(x-y)u(y,t)dy - \gamma'(x)X(t) \tag{59}$$

$$u_x(x,t) = w_x(x,t) + \int_0^x \Psi_x(x-y)w(y,t)dy + \psi'(x)X(t) \tag{60}$$

it can be shown that

$$\|w_x\|^2 \leq \alpha_3\|u_x\|^2 + \alpha_4\|u\|^2 + \alpha_5|X|^2 \tag{61}$$

$$\|u_x\|^2 \leq \beta_3\|w_x\|^2 + \beta_4\|w\|^2 + \beta_5|X|^2 \tag{62}$$

where $\alpha_3 = 3$, $\alpha_4 = 3\|\Gamma\|^2$, $\alpha_5 = 3\|\gamma'\|^2$, $\beta_3 = 3$, $\beta_4 = 3\|\Psi\|^2$, and $\beta_5 = 3\|\psi'\|^2$.

Similarly, for the direct and inverse backstepping transformations

$$z(x,t) = w(x,t) - \int_0^x \kappa(x,y)w(y,t)dy \tag{63}$$

$$w(x,t) = z(x,t) + \int_0^x p(x,y)z(y,t)dy \tag{64}$$

$$z_x(x,t) = w_x(x,t) - \int_0^x \kappa'(x,y)w(y,t)dy \tag{65}$$

$$w_x(x,t) = z_x(x,t) + \int_0^x p'(x,y)z(y,t)dy \tag{66}$$

we obtain that

$$\|z\|^2 \leq \alpha_6\|w\|^2 \tag{67}$$

$$\|w\|^2 \leq \beta_6\|z\|^2 \tag{68}$$

$$\|z_x\|^2 \leq \alpha_7\|w_x\|^2 + \alpha_8\|w\|^2 \tag{69}$$

$$\|w_x\|^2 \leq \beta_7\|z_x\|^2 + \beta_8\|z\|^2 \tag{70}$$

where $\alpha_6 = 2(1+\|\kappa\|^2)$, $\beta_6 = 2(1+\|p\|^2)$, $\alpha_7 = 2$, $\alpha_8 = 2\|\kappa'\|^2$, $\beta_7 = 2$ and $\beta_8 = 2\|p'\|^2$.

Combining (57)-(58), (61)-(62) and (67)-(70), we have

$$\|u\|^2 \leq \beta_1\beta_3\|z\|^2 + \beta_2|X|^2 \tag{71}$$

$$\|z\|^2 \leq \alpha_1\alpha_3\|u\|^2 + \alpha_2\alpha_3|X|^2 \tag{72}$$

$$\|u_x\|^2 \leq \beta_3\beta_7\|z_x\|^2 + (\beta_3\beta_8 + \beta_4\beta_6)\|z\|^2 + \beta_5|X|^2 \tag{73}$$

$$\|z_x\|^2 \leq \alpha_3\alpha_7\|u_x\|^2 + (\alpha_4\alpha_7 + \alpha_1\alpha_8)\|u\|^2 + (\alpha_5\alpha_7 + \alpha_2\alpha_8)|X|^2 \tag{74}$$

Consider the Lyapunov function candidate

$$V(t) = \overline{X}^T PX + \frac{a}{2}||z(x,t)||^2 + \frac{b}{2}||z_x(x,t)||^2 \tag{75}$$

where the matrix $P = P^T > 0$ is the solution to the Laypunov equation

$$P(A+BK) + \overline{(A+BK)}^T P = -Q$$

for some $Q = Q^T > 0$, and the parameters $a, b > 0$ are to be chosen later.

According to (71)-(74), we have

$$\underline{\delta}(|X|^2 + \|u\|^2 + \|u_x\|^2) \leq V \leq \bar{\delta}(|X|^2 + \|u\|^2 + \|u_x\|^2)$$

where

$$\underline{\delta} = \frac{\min\{\lambda_{\min}(P), \frac{a}{2}, \frac{1}{2}\}}{\max\{1+\beta_2+\beta_5, \beta_1\beta_3 + \beta_3\beta_8 + \beta_4\beta_6, \beta_3\beta_7\}} \tag{76}$$

$$\bar{\delta} = \max\{\lambda_{\max}(P) + \frac{a}{2}\alpha_2\alpha_3 + \frac{1}{2}(\alpha_5\alpha_7 + \alpha_2\alpha_8), \frac{a}{2}\alpha_1\alpha_3 + \frac{1}{2}(\alpha_4\alpha_7 + \alpha_1\alpha_8), \frac{1}{2}\alpha_3\alpha_7\} \tag{77}$$

Taking a derivative of the Lyapunov function (75) along the solutions of the PDE-ODE system (5)-(8), we get

$$\begin{aligned}\dot{V} &= -\overline{X}^T QX + 2\overline{X}^T PBz(0,t) + \frac{a}{2}\int_0^1 \dot{z}\overline{z}dx \\ &\quad + \frac{a}{2}\int_0^1 z\dot{\overline{z}}dx + \frac{b}{2}\int_0^1 \dot{z}_x\overline{z}_x dx + \frac{b}{2}\int_0^1 z_x\dot{\overline{z}}_x dx \\ &= -\overline{X}^T QX + 2\overline{X}^T PBz(0,t) \\ &\quad -ac\int_0^1 z\overline{z}dx - bc\int_0^1 z_x\overline{z}_x dx \\ &\leq -\frac{\lambda_{\min}(Q)}{2}|X|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)}z^2(0,t) \\ &\quad -ac\int_0^1 z\overline{z}dx - bc\int_0^1 z_x\overline{z}_x dx\end{aligned} \tag{78}$$

Noting $z(1,t) = 0$, applying Poincar$\acute{e}$'s inequality and Agmon's inequality, we have

$$\int_0^1 z\overline{z}dx \leq 4\int_0^1 z_x\overline{z}_x dx \tag{79}$$

$$z^2(0,t) \leq \max_{0\leq x\leq 1} z^2(x,t) \leq 2\|z(t)\|\|z_x(t)\| \tag{80}$$

which results in

$$z^2(0,t) \leq 4\int_0^1 z_x\overline{z}_x dx \tag{81}$$

Substituting (81) into (82), we have

$$\dot{V} \leq -\frac{\lambda_{\min}(Q)}{2}|X|^2 - ac\int_0^1 z\overline{z}dx - \left(bc - \frac{8|PB|^2}{\lambda_{\min}(Q)}\right)\int_0^1 z_x\overline{z}_x dx \tag{82}$$

Taking

$$bc > \frac{8|PB|^2}{\lambda_{\min}(Q)} \tag{83}$$

we get

$$\dot{V} \leq -dV \tag{84}$$

where

$$d = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, 2c - \frac{16|PB|^2}{b\lambda_{\min}(Q)}\right\}. \tag{85}$$

Hence,

$$\begin{aligned} & |X(t)|^2 + \|u(x,t)\|^2 + \|u_x(x,t)\|^2 \\ \leq \; & \frac{\bar{\delta}}{\underline{\delta}} e^{-dt}\left(|X(0)|^2 + \|u(x,0)\|^2 + \|u_x(x,0)\|^2\right) \end{aligned} \tag{86}$$

for all $t > 0$, which completes the proof. ■

IV. Simulation Results

In this section, we present the results of numerical simulations for a scalar case of the system (1)-(4), where we choose $A = 1 + i$, $B = 2i$, $C = i$. And K is chosen as $K = 3i$ such that $A + BK = -5 + i$ is Hurwitz. The design parameter c in the target system (5)-(8) is chosen as 4. The response of the system is shown in Figs. 3 and 4, where we can observe that the closed-loop system is exponentially stable. Fig. 5 shows the control effort.

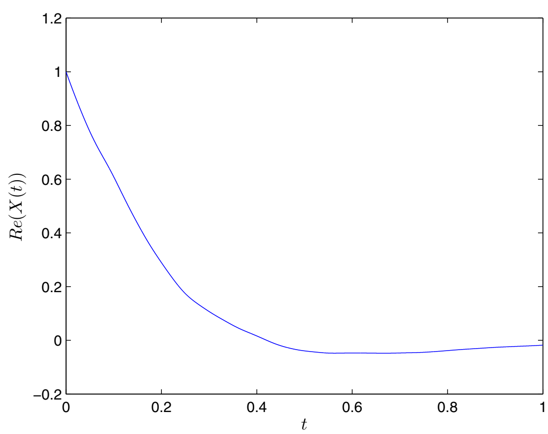

Fig. 3. The response of the ODE. Only the real part is shown.

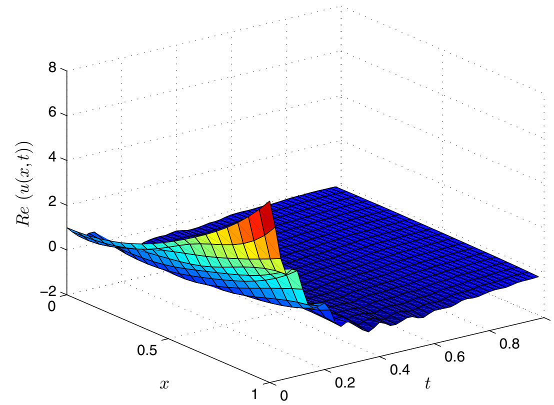

Fig. 4. The response of the Schrodinger equation. Only the real part is shown.

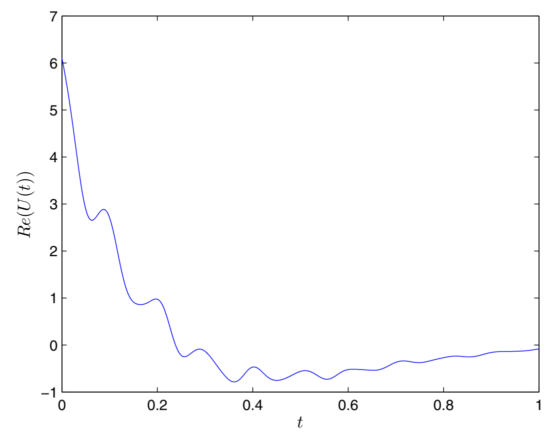

Fig. 5. Control effort. Only the real part is shown.

V. Conclusion

In this paper, we have developed a two-step backstepping control design for stabilization of an ODE and Schrödinger cascade by introducing an intermediate target system and an intermediate control. The exponential stability of the closed-loop system has been investigated using the Lyapunov method. The effectiveness of our controller design has been illustrated by a numerical example.

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