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Robust Tracking Control for Underactuated Autonomous Vehicles Using Feedback Linearization

D.H.S. Maithripala and J. M. Berg

Abstract—We present a feedback linearization control for output-feedback tracking of a class of underactuated vehicles, with guaranteed stable zero dynamics, zero steady-state tracking error in the presence of constant disturbance forces and moments, and significant robustness to parameter variations. The potential benefits of the approach are illustrated with simulations.

I. INTRODUCTION

Center of mass trajectory tracking of under-actuated mechanical systems are of great interest due to its wide application in autonomous vehicles [1], [2]. A brief summary of the existing control methods for underactuated systems is provided in [3], [2], [4], [5]. The area has witnessed a surge of activity owing to recent developments in the field of quadrotor device technology and their wide potential applications in surveillance, exploration, mapping, and cooperative control [6]. The quadrotor is a 3D rigid body actuated by one degree of freedom in the translational direction and fully actuated in the rotational degrees of freedom. This is a common trait in many vehicular systems such as differential wheeled mobile robots, hovercrafts, VTOLs, submersibles, surface vessels, and aircraft.

Many quaternion based tracking controllers exist in the literature and typically suffer from complexities that result due to the ambiguity of the 2 to 1 covering of SO(3) by the space of unit quaternions [7]. Another class of approaches used in the literature are based on Euler angle parameterization of the attitude such as the one used in [8]. A weakness of such methods is also the non-global convergence of the controllers due to the coordinate singularities inherent in Euler angles [7].

Back-stepping based ideas have been successfully used for the center of mass tracking of a point [2], [4] almost globally and robustly. An intrinsic adaptive switching supervisory control in conjunction with back-stepping has been developed in [2] that ensures that the tracking error is globally bounded and converges to a neighborhood of the origin in the presence of large parametric uncertainty. The size of the neighborhood can be made arbitrarily small by choosing the gains appropriately. The robust and intrinsic tracking controller developed in [4] guarantees almost global convergence in the absence of un-modeled errors. However in the presence of uncertainties the tracking error is only guaranteed to be uniformly bounded with a local region of attraction where the size of the bound can be made arbitrarily small and the region of attraction can be made large by choosing high gains. A quaternion based controller is developed by [3] and also follows a two step back-stepping technique. The advantage of the controller is that it does not require velocity measurements and the convergence is guaranteed globally. A robustness analysis of the results are not available.

In contrast to the work described above and the references therein, we consider a generalized version of the tracking problem. The general problem is to enforce a rigid body to move in such a way that its center of mass appears at a prescribed desired position with respect to a moving reference frame. We allow provision for the prescribed point to change. To the best of our knowledge this generalized tracking problem has not been solved in the literature. The problem that is solved by all [2], [4], [3], [5], [8], [6] and the references therein belong to the special case where the prescribed point coincides with the center of the reference frame. Solving the general problem defined above has significant practical implications for formation control [9], [10], [11]. We will demonstrate this usefulness using simulations involving a group of quadrotor UAVs.

The major contribution of this paper is to show how to solve this generalized tracking problem with almost global convergence in the absence of parametric uncertainties and un-modeled constant inertial forces and body moments. The convergence will be shown to be global for the special case, that corresponds to the tracking problem considered in the literature so far, where the center of the mass of the body is forced to track the origin of the reference frame.

The controller approach we use is a two step process. First we design a globally convergent controller that will ensure that the system output tracks the output of a suitably chosen fully actuated exogenous mechanical system. In the next step we design an almost globally convergent tracking controller for the exogenous system so that the output of the exogenous system tracks a given sufficiently smooth reference output in the absence of uncertainties. This second step of the process is always possible [12], [13]. In the special case where the center of mass of the object tracks the origin of the reference frame the second step of the process in fact yields a globally convergent controller rendering the composite scheme globally convergent in the absence of parametric uncertainties and in the presence of small constant uncertain inertial forces and body moments. We will show that the controller is robust with respect to parametric uncertainties and un-modeled constant inertial forces and body moments. The tracking error can be made to converge to an arbitrarily small neighborhood of the

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origin with an arbitrarily large region of attraction by selecting high gains for possibly large uncertainties.

We use feedback linearization techniques to solve the first step. Feedback linearization techniques offer a simpler approach to the design of tracking controllers. A serious drawback of these methods is that they are known to fail for under actuated systems due to either the loss of invertibility or due to the presence of unstable zero dynamics [2], [5]. In this work we show that both these obstacles can be overcome. We feel that this is also a major contribution of the paper since traditionally feedback linearization methods are considered to have poor robustness properties. The results presented apply for a wide class of 2D and 3D under actuated rigid body systems that include differential wheeled mobile robots, hovercraft, quadrotors, VTOLs, underwater and surface vessels, and aircraft.

The rest of the paper is organized as follows. In Section II we present the formulation and the solution procedure of the general problem. We explicitly show how to construct this controller for 3D rigid body tracking in Section III. The developed controller is demonstrated on a 3D quadrotor in Section IV. Simulation results show excellent convergent properties in the presence of large parameter uncertainties and unmodeled piecewise constant forces. We simulate the tracking controller in conjunction with the decentralized formation control approach developed in [10], [11] to demonstrate how a group of 3D quadrotors arrive at and maintain rigid formation, while a prescribed member tracks a desired reference.

II. UNDERACTUATED RIGID BODY MOTION

We consider rigid body motion in two or three dimensions, which we write \( \mathbb{R}^d \), where \( d \) is either 2 or 3. The corresponding configuration space is the special Euclidean group \( SE(d) \approx SO(d) \times \mathbb{R}^d \). The control objective is to control a rigid vehicle, denoted \( A \), so that its center of mass tracks a desired trajectory. The point \( b_r(t) \) specifies the desired center of mass location relative to a virtual structure (VS) or “exosystem,” denoted \( V \). Let \( (R_r(t), o_r(t)) \) denote the desired reference of the VS.

Let \( q_A = (\bar{R}_A, o_A) \in SO(d) \times \mathbb{R}^d \) denote a particular configuration of the rigid body. The body angular velocity is \( \Omega_A = \bar{R}_A \dot{\bar{R}}_A \), and the body center of mass is expressed in the body frame as \( V_A = \bar{R}_A^T \hat{0}_A \). The kinetic energy is \( KE = (I_A \Omega_A \cdot \Omega_A + M_A ||V_A||^2)/2 \) where \( I_A \) is the inertia tensor of the body and \( M_A \) is the total mass of the body. Let \( \tau_A = I_A (\bar{T}_A^a + \Delta_T) \) be the moments acting on the body expressed in the body frame, and let \( f_A = M_A (\chi_A^a + \hat{\chi}_A^a + R^A_\Delta \chi) \) be the forces acting on the body expressed in the body frame. The control moment is denoted by \( I_A \bar{T}_A^a \) and the control force is denoted by \( M_A \chi_A^a \). The force \( M_A \chi_A^a \) represents modeled constraint forces such as gravitational and nonholonomic forces. The body moment \( I_A \Delta_T \) and the inertial force \( M_A \chi_A^a \) represent unmodeled moments and forces, which are assumed to be constant.

We assume that the rigid body \( A \) is fully actuated in the rotational degrees of freedom, and is actuated only in one of the translational degrees of freedom. That is, \( \dim(\text{span}\{\chi_A^a\}) = 1 \).

In this paper we consider the generalized tracking problem of enforcing movement of a rigid body \( A \) in such a way that its center of mass appears fixed at \( b_r(t) \) with respect to the reference frame with configuration \( q_r(t) = (R_r(t), o_r(t)) \). Thus let

\[
y(t) \triangleq h(q_A(t), q_r(t)) = o_A(t) - o_r(t) - R_r(t)\dot{b}_r(t). \tag{1}
\]

Given a reference \( q_r(t) = (R_r(t), o_r(t), b_r(t)) \in SO(d) \times \mathbb{R}^d \times \mathbb{R}^d \) the tracking objective is to find control the forces and control torques on the rigid body, \( (\chi_A^a, T_A^a) \), such that \( \lim_{t \to \infty} y(t) \to 0 \) for a given desired \( q_r(t) \). Here \( q_r(t) \) is a piecewise defined sufficiently smooth reference. To the best of our knowledge this generalized tracking problem has not been solved in the literature. The problem that is solved in the literature belong to the special case where \( b_r(t) \equiv 0 \).

Solving the general problem defined above has significant practical implications for formation control [9], [10], [11]. We demonstrate this versatility of this formulation on simulations of a group of quadrotor UAVs.

In the following we will show how to solve the generalized tracking problem with almost global convergence in the presence of small parametric uncertainties and unmodeled constant step type forcing. The convergence will be shown to be global for the special case where \( b_r(t) \equiv 0 \) that corresponds to the tracking problem considered in the literature so far. We solve the general problem stated above in two steps. In the first step we design a globally convergent robust tracking controller that will ensure \( \lim_{t \to \infty} h(q_A(t), q_V(t)) \to 0 \) where \( q_V(t) \in Q_V \) is the configuration of a exogenous fully actuated unconstrained mechanical system defined on the configuration space \( Q_V = SO(d) \times \mathbb{R}^d \times \mathbb{R}^d \). In the special case where \( b_r(t) \equiv 0 \) the exosystem configuration space is \( Q_V = \mathbb{R}^d \).

In the next step we derive a configuration tracking controller that will drive the exosystem configuration \( q_V(t) \) to \( q_r(t) \) asymptotically. It is then clear that both controllers in combination will solve the generalized tracking problem. Since the exosystem is fully actuated and the output space is \( Q_V = SO(d) \times \mathbb{R}^d \times \mathbb{R}^d \) it follows from the results of [12], [13] that a controller exists that solves the second step almost globally for piecewise defined sufficiently smooth configuration references. In the special \( Q_V = \mathbb{R}^d \) the convergence is global. The major contribution of the paper is to show that the first step can be solved.

We now state the above problem formally. Consider the augmented system

\[
\dot{\bar{R}}_A = R_A \dot{\bar{\Omega}}_A, \tag{2}
\]
\[
\dot{\bar{\Omega}}_A = R_A \bar{V}_A, \tag{3}
\]
\[
\bar{V}_A = -\bar{\Omega}_A \bar{V}_A + \bar{\chi}_A^a + \hat{\bar{\chi}}_A^a + R^A_\Delta \chi, \tag{4}
\]
\[
\hat{\bar{\Omega}}_A = I_A^{-1} (I_A \bar{\Omega}_A \times \bar{\Omega}_A) + T_A^a + \Delta_T, \tag{5}
\]
\[
\dot{R}_V = R_V \bar{v}_V, \tag{6}
\]
\[
\dot{\bar{v}}_V = -\bar{\Omega}_V \bar{v}_V + \bar{\chi}_V, \tag{7}
\]
\[
\hat{\bar{\Omega}}_V = I_V^{-1} (I_V \Omega_V \times \Omega_V) + V_V, \tag{8}
\]
\[
\dot{\bar{v}}_V = \alpha_V. \tag{9}
\]
Here equations (2) — (5) represent the 3D rigid body motion of the rigid body of interest $\mathcal{A}$, and the exosystem is described by the equations (6)–(10). The equations (6)–(9) represent a 3D rigid body Euclidean motion while the equation (10) represents the deformation of a certain given point $P$ with respect to the body $\mathcal{V}$. The exosystem is assumed to be fully actuated and hence $\dim(\text{span}\{ T_V \}) = n$, $\dim(\text{span}\{ \chi_V \}) = d$, and $\dim(\text{span}\{ \alpha_V \}) = d$.

Remark 1. We note that in the case where $b_r(t) \equiv 0$ the exosystem is given by the single particle system on $\mathcal{Q}_V = \mathbb{R}^d$ given by

$$\dot{\alpha}_V = V_V, \quad (11)$$
$$\dot{V}_V = \chi_V. \quad (12)$$

Let $\Omega_E(t) = E - E^T$ where $E(t) = R_r(t)R_E^T (t)$. Then from the results of [12], [13] it easily follows that \(\lim_{t \to \infty} (R_r(t), \alpha_V(t), b_V(t)) \to (R_r(\infty), \alpha_V(\infty), b_V(\infty))\) almost globally if the equations forces are selected such that

$$\chi_{VIa} = R_V^T (\alpha_r(t) - k_r (\dot{\alpha}_V(t) - \dot{\alpha}_r(t)) - k_p (\alpha_V(t) - \alpha_r(t))), \quad (13)$$
$$T_{Va} = R_V^T \Omega_E(t) - k_r \alpha_V(t), \quad (14)$$
$$\alpha_{Va} = b_r(t) - k_r (b_V(t) - \bar{b}_r(t)) - k_p (b_V(t) - b_r(t)). \quad (15)$$

Remark 2. In the case where $b_r(t) \equiv 0$ the controller given by

$$\chi_{Va} = \dot{\alpha}_r(t) - k_r (\dot{\alpha}_V(t) - \dot{\alpha}_r(t)) - k_p (\alpha_V(t) - \alpha_r(t)), \quad (16)$$

guarantees $\lim_{t \to \infty} \alpha_V(t) \to \alpha_r(t)$ globally.

We now show how output feedback control techniques can be applied in a systematic and intrinsic fashion to solve the first step of the controller design that will ensure that $\lim_{t \to \infty} h(q_A(t), q_V(t)) \to 0$ globally in the presence of small parameter uncertainties and un-modeled step forces of finite duration. To do so we repeatedly differentiate $y = o_A - o_V - R_V V_V$ to obtain

$$\dot{y} = R_A (V_A(t) - R_V (V_V(t) + \dot{\chi}_V(t)) b_V(t) + b_V(t)). \quad (17)$$
$$\ddot{y} = R_A (\chi_A + \dot{\chi}_A + R_A^T \Delta x) - \gamma_V(t), \quad (18)$$
$$\dddot{y} = R_A \left( \Omega_A (\chi_A + \dot{\chi}_A) + (\chi_A + \dot{\chi}_A) \right) - \gamma_V(t), \quad (19)$$
$$\dddot{\gamma}_V = R_A \left( \Omega_A (\chi_A + \dot{\gamma}_V(t)) + 2 \Omega_A (\chi_A + \dot{\chi}_A) \right) + (\chi_A + \dot{\gamma}_V(t)) - (\chi_A + \dot{\chi}_A) \times \Omega_A - \gamma_V(t), \quad (20)$$

where $\gamma_V(t) = R_V (V_V(t) + \dot{\chi}_V(t)) b_V(t) + b_V(t)).$

Remark 3. In the case where $b_r(t) \equiv 0$ we see that $\gamma_V(t) = V_V(t)$.

From these expressions we conclude the following:

I. Rigid bodies fully actuated in the translational degrees of freedom are relative degree two. Furthermore the zero dynamics when the system is constrained to $y \equiv 0$ and \(\dot{y} \equiv 0\) are bounded are given by the exosystem dynamics and the rotational dynamics of the rigid body. The moments are fully available to independently almost globally stabilize the attitude of the rigid body to any desired attitude. Thus the zero dynamics of the system are guaranteed to be bounded if the exosystem states are bounded. The control (13)–(15) ensures that the exosystem states are bounded if the reference is a piecewise defined twice differentiable bounded trajectory.

II. For differential wheeled mobile robots that can be modeled using the unicycle model the no-slip constraint implies that the constraint force $\chi_A^c$ is a function of $\Omega_A$ and hence the system is relative degree three. The zero dynamics given by $y \equiv 0$, $\dot{y} \equiv 0$, and $\ddot{y} \equiv 0$ are bounded if the exosystem states are bounded. The control (13)–(15) ensures that the exosystem states are bounded if the reference is a piecewise defined thrice differentiable bounded trajectory.

III. For constrained rigid body systems where $\chi_A^c$ is not a function of $\Omega_A$ the system is relative degree four when $||\chi_A^c|| \neq 0$. This includes center of mass tracking of hovercrafts, planar quadrotors (and PV-TOLs), 3D quad rotors, aircrafts, surface vessels, and underwater vehicles. For PV-TOLs, and the hovercraft the zero dynamics are bounded if the exosystem states are bounded. The control (13)–(15) ensures that the exosystem states are bounded if the reference is a piecewise defined four times differentiable bounded trajectory. Below we show how to pick additional degrees of freedom available in the actuation of the rotational agrees of freedom to ensure that the zero dynamics are bounded for the 3D quadrotor.

The constant relative degree condition and the fact that the zero dynamics are either stable or can be stabilized implies the existence of a dynamic feedback linearizing controller that will solve the first step of ensuring $\lim_{t \to \infty} h(q_A(t), q_V(t)) \to 0$ globally. In the next section we show explicitly how to construct this controller for the case of 3D rigid body motions that include the quad rotor as a special case of interest.

III. 3D UNDERACTUATED TRACKING

To illustrate the ideas discussed above we consider 3D motion. The behavior of the system is described by the equations (2)–(5) and the exosystem behavior is described by the equation (6)–(10). We assume without loss of generality that $\chi_A^c$ expressed in the body frame is $\chi_A^c = [0 \quad 0 \quad \frac{f_{A2}}{M_A}]$ and $\chi_A^c = -g R_3^T c_3$ where $c_3 = [0 \quad 0 \quad 1]^T$. Let $T_A = [\frac{\tau_{A1}}{T_{A1}} \frac{\tau_{A2}}{T_{A2}} \frac{\tau_{A3}}{T_{A3}}]$. The inputs to the system are $(f_{A1}, \tau_{A1}, \tau_{A2}, \tau_{A3})$. Let $\chi_V$ expressed in the body frame be $\chi_V = [\frac{f_{V1}}{M_V} \frac{f_{V2}}{M_V} \frac{f_{V3}}{M_V}]$. We note that these are exactly the rigid body equations that describe the motion of quad rotors, VTOLs, aircrafts, and underwater vehicles.

From (20) we observe that the system is relative degree four with dynamic extension with respect to output $y$ and input $(f_{A1}, \tau_{A1}, \tau_{A2})$. Notice that $\tau_{A3}$ is not utilized in the dynamic feedback control. The dynamic output feedback linearizing control is given by

$$\begin{align*}
\tau_{A1} &= \alpha (q_A, v_A, q_V, v_V, f_{A1}, \dot{f}_{A3}, \chi_V, \dot{\chi}_V, \ddot{\chi}_V), \quad (21) \\
\dot{\chi}_V &= \Phi_V, \quad (22) \\
\ddot{f}_{A3} &= \Phi_A, \quad (23)
\end{align*}$$

with (21) being the solution to

$$\begin{align*}
R_A \left( \Omega_A^2 (\chi_A + \dot{\gamma}_V) + 2 \Omega_A (\chi_A + \dot{\chi}_A) \right).
\end{align*}$$
where the constants $a_0, a_1, a_2, a_3$ are such that $(s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)$ is a Hurwitz polynomial.

The zero dynamics of the system are given by the dynamics of the system constrained to $y \equiv 0$, $\dot{y} \equiv 0$, and $\ddot{y} \equiv 0$. Thus from (1) and (17)–(19) we see that when the dynamics are reduced to the zero dynamics $\alpha, V_A, \Omega_{A1}$, and $\Omega_{A2}$ can be explicitly solved in terms of $R_A$, and the exosystem states. Since $R_A$ is always bounded, the above system variables will remain bounded if the exosystem states remain bounded. What remains is to ensure that $\Omega_{A3}$ is bounded. It easily follows that the control

$$\tau_{A3} = -k_{13} \Omega_{A3},$$

asymptotically enforces the constraint, $\Omega_{A3} = 0$. With this auxiliary constraint we have that the zero dynamics are bounded if the reference is a piece wise defined four times differentiable bounded reference. In the case where no model uncertainties exist the closed loop dynamics are given by the linear dynamics $y_I^{(5)} + a_4 y_I^{(4)} + a_3 y_I^{(3)} + a_2 y_I^{(2)} + a_1 y_I^{(1)} + a_0 y_I = 0$ where $y_I = \int y dt$ and $y_I^{(n)}$ denotes the $n^{th}$ time derivative of $y_I(t)$. Since the polynomial $(s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)$ is Hurwitz we conclude that the controller (21)–(23) and (25) with (13)–(15) and

$$\Phi_v = \begin{bmatrix} \chi y_a - k_1 (\chi y_a - \chi y_a^2) - k_2 (\chi y_a^2 - \chi y_a) \\ T y_a - k_1 (T y_a - T y_a^2) - k_2 (T y_a^2 - T y_a) \\ \alpha y_a - k_1 (\alpha y_a - \alpha y_a^2) - k_2 (\alpha y_a^2 - \alpha y_a) \end{bmatrix}$$

ensures that $\lim_{t \to \infty} \alpha(t) \to \alpha_\tau(t) + R_{\alpha}(t)b(t)$ in the absence of model uncertainties. Thus we have proved the following lemma:

**Lemma 1.** In the absence of model uncertainties the controls given by (13)–(15), (21)–(23), (25) and (26) ensures that $\lim_{t \to \infty} y(t) \to 0$ almost globally, where $y(t)$ is given by (1), for any piece wise defined four times differentiable bounded reference $(R_{\alpha}(t), \alpha(t), b(t))$.

From remark-1 and remark-2 we easily obtain the following corollary:

**Corollary 1.** For the special case where $b(t) \equiv 0$, the controls given by (13)–(15), (21)–(23), (25) and (26) ensure that $\lim_{t \to \infty} \alpha(t) \to \alpha_\tau(t)$ globally for any piece wise defined four times differentiable bounded reference $\alpha_\tau(t)$ in the absence of model uncertainties

We will now show that the controller ensures that the tracking error converges to a neighborhood of the origin with a large region of attraction in the presence of bounded un-modeled parameter uncertainties and un-modeled constant inertial forces and body moments, $\Delta \chi$ and $\Delta v$. We will also show that the tracking error can be made arbitrarily small and the region of attraction can be made arbitrarily large by choosing high gains.

Define $\epsilon_M = (1 - M_0 / M), \epsilon_I = (I_{3 \times 3} - I - I_0), z \equiv [y_I \ y_I^{(1)} \ y_I^{(2)} \ y_I^{(3)} \ y_I^{(4)}]_T, \Delta \phi = [\Delta \chi \ \Delta \psi]_T$ where $M_0$ and $I_0$ are the nominal inertia properties. Then one can show that the closed loop dynamics reduce to the form

$$\dot{z} = A(\epsilon_M) z + B_3(\gamma(t) - \Delta \chi, \Delta \phi) + B_{12}(z, \epsilon_I) + B_{13}(z, \epsilon_I) + B_{14}(z, \epsilon_I) + B_{24}(z, \Delta \phi) + \epsilon_M \eta(\Delta \chi, \Delta \phi, \gamma(t), \gamma(t), \gamma(t)).$$

The matrix $A(\epsilon_M)$ is a constant $15 \times 15$ block canonical matrix where $A(0)$ is Hurwitz with characteristic polynomial $(s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)$. The matrix $\eta$ satisfies $\eta(0,0, \gamma(t), \gamma(t), \gamma(t)) \equiv 0$ and $\eta \leq \alpha_\eta$ for some $\alpha_\eta > 0$. The matrices $B_{13}(\gamma(t) - \Delta \chi, \epsilon_I, \epsilon_I), B_{12}, B_{13}$ and $B_{14}$ are $15 \times 1$ matrices that satisfy the following conditions:

$$\|B_3(\gamma(t) - \Delta \chi, \Delta \phi)\| \leq \|\Delta \phi\| \|\gamma(t) - \Delta \chi\| \quad \forall \ z,$$

$$\|B_{12}(z, \epsilon_I)\| < \beta_{\epsilon_1} \|\epsilon_I\| \|\gamma(t) - \Delta \chi\| \quad \forall \ z,$$

$$\|B_{13}(z, \epsilon_I)\| < \beta_{\epsilon_2} \|\epsilon_I\| \|\gamma(t) - \Delta \chi\| \quad \forall \ z,$$

In each of these matrices only the last three rows are non-zero. Using these properties we can prove the following theorem.

**Theorem 1.** In the presence of bounded parametric uncertainties and un-modeled constant inertial forces and body moments the controls given by (13)–(15), (21)–(23), (25) and (26) ensures that tracking error of the generalized tracking problem converges to a neighborhood of the origin for a large set of initial conditions. The controller gains can be selected such that the error is arbitrarily small and the region of attraction is arbitrarily large.

**IV. SIMULATION RESULTS FOR QUAD ROTOR TRACKING**

Simulation results shown in Figures 1–4 demonstrate the effectiveness of this controller for trajectory tracking in the absence of parametric uncertainties but in the presence of un-modeled constant forces.

We have chosen $R_{\epsilon}(t) = I_{3 \times 3}, b(t) = [0 \ 0 \ 0]^T$, and

$$\alpha_{\tau}(t) = 50 \sin(t) \cos(t) \cos(t + \pi/4)^T.$$  

This defines a circular reference trajectory for the quad rotor to track.

The nominal values of the mass and the inertia matrix used in the controller design were chosen to be $M_0 = 1$ and $I_0 = \text{diag}\{1, 1, 2\}$ while the actual mass and the inertia matrix of the quad rotor were chosen to be $M = 2M_0$ and $I = 2I_0$ respectively. We assume that a large unknown inertial force of $\Delta \chi = [2g; 0; 0]$ representing a piecewise constant wind gust in the $\epsilon_1$ inertial direction and a constant body moment of $\Delta \phi = [2; 2; 2]$ is acting on the quadrotor.

The initial conditions chosen for the quad rotor are, $R_A(0) = \text{diag}\{1, -1, -1\}, \omega_A(0) = [-50, -50, -50]^T, \Omega_A(0) = [0, 0, 0]^T, V_A(0) = [0, 0, 0]^T, \chi_A = [0, 0, 1.5g]^T$, and $\chi_{\Delta} = [0, 0, 0]^T$. The gravitational constant $g$ is chosen to be 1. We have chosen the initial configuration to correspond to an upside down configuration in order to demonstrate the versatility of the controller. The controller gains are chosen to be $k_1 = k_2 = 10, k_v = k_p = 0.5$, with
Fig. 1. A quadrotor tracking a circular trajectory starting from an initially upside down configuration in the absence of parametric uncertainty but in the presence of constant unmodeled forcing.

Fig. 2. The tracking error for a quadrotor tracking a circular trajectory starting from an initially upside down configuration in the absence of parametric uncertainty but in the presence of constant unmodeled forcing.

\[ a_0, a_1, a_2, a_3, a_4 \] chosen to be such that 
\[ (s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0) = (s + 5)^5. \]

Figure 3 and Fig. 2 show excellent convergence in the presence of large parameter uncertainties and unmodeled piecewise constant forces.

Finally we apply the tracking controller in conjunction with the decentralized formation control approach developed in [10], [11] to ensure that a group of seven 3D quadrotors arrive at formation and maintain rigid formation while the agent at the center of the formation tracks a circle in 3D space. Simulation results are shown in Fig. 5 and Fig. 6.

V. CONCLUSION

This paper presents an intrinsic output-feedback based controller for center of mass tracking for a class of underactuated rigid body vehicle dynamics. Specifically, the vehicle rotational dynamics are assumed to be fully actuated, while the vehicle translational dynamics are assumed to have a single control input. The system may be subject to nonholonomic constraints. The tracking problem that we have considered is a general one, where a rigid body is enforced to move in such a way that its center of mass appears at a—possibly time-varying—point with respect to a moving reference frame. This includes conventional center of mass tracking of a given trajectory as a special case. In the absence of parameter uncertainties and moment disturbances, the controller guarantees almost-global asymptotic stability of the tracking error, with global asymptotic stability if the tracking point is the origin of the moving frame. The controller is robust to parameter uncertainty and constant moment disturbance in the sense that for given bounds on the parameter uncertainty, and the moment disturbance the maximum tracking error can be made arbitrarily small, and the region of convergence may be made arbitrarily large. The method is illustrated on a 3D quadrotor model, and demonstrated on simulations of single and multiple quadrotors in 3D.

REFERENCES

We sketch the proof of Lemma-1 and Theorem-1.

**Proof.** Let $W = z^TPz$ where $ATP + PA = -Q$ with $Q > 0$. Then for $||z|| \geq 1$

$$W = -z^TQz + 2z^TP(A_\gamma + B_\epsilon + B_\Delta) \leq -\lambda_{\text{min}}(Q)||z||^2 + 2||z||^2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + \beta_\epsilon_1 ||\dot{\gamma} - \Delta_\chi||||\epsilon_I|| + \beta_\epsilon_2 ||\epsilon_I||||z||^2
$$

$$+ ||\Delta_T||||z|| + \epsilon_M||\eta||)^2 \leq (\lambda_{\text{min}}(Q) + 2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + ||\Delta_T|| + \beta_\epsilon_1 ||\dot{\gamma} - \Delta_\chi||||\epsilon_I|| + \epsilon_M||\eta||)||z||^2
$$

$$+ 2\beta_\epsilon_2 ||P||||\epsilon_I||||z||^3 \leq (\lambda_{\text{min}}(Q) + 2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + ||\Delta_T|| + \beta_\epsilon_1 ||\dot{\gamma} - \Delta_\chi||||\epsilon_I|| + \epsilon_M||\eta||)) / 2\beta_\epsilon_2 ||P||||\epsilon_I||
$$

Thus $W$ is decreasing if

$$||z|| < (\lambda_{\text{min}}(Q) - 2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + ||\Delta_T|| + \beta_\epsilon_1 ||\dot{\gamma} - \Delta_\chi||||\epsilon_I|| + \epsilon_M||\eta||)) / 2\beta_\epsilon_2 ||P||||\epsilon_I||
$$

The error is globally bounded if there are no parametric uncertainties. Likewise for $||z|| \leq 1$ we find that $W$ is decreasing if

$$||z|| > 2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + \beta_\epsilon_1 ||\dot{\gamma} - \Delta_\chi||||\epsilon_I|| + \beta_\epsilon_2 ||\epsilon_I|| + ||\Delta_T|| + \epsilon_M||\eta||) / \lambda_{\text{min}}(Q)
$$

and so we have that $W$ is decreasing in the region

$$2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + \beta_\epsilon_1 ||\dot{\gamma} - \Delta_\chi||||\epsilon_I|| + \beta_\epsilon_2 ||\epsilon_I|| + ||\Delta_T|| + \epsilon_M||\eta||) / \lambda_{\text{min}}(Q)
$$

$$< ||z|| < (\lambda_{\text{min}}(Q) - 2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + ||\Delta_T|| + \epsilon_M||\eta||)) / 2\beta_\epsilon_2 ||P||||\epsilon_I||.
$$

Therefore

$$\lim_{t \to \infty} ||z(t)|| \to 2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + \beta_\epsilon_1 ||\dot{\gamma} - \Delta_\chi||||\epsilon_I|| + \beta_\epsilon_2 ||\epsilon_I|| + ||\Delta_T|| + \epsilon_M||\eta||) / \lambda_{\text{min}}(Q).
$$

From this we can show that we can make the region of convergence arbitrarily large and the error bound arbitrarily small by choosing the gains to be sufficiently large. In the absence of parametric uncertainty, globally,

$$\lim_{t \to \infty} ||z(t)|| \to 2||P||(||\Delta_T||||\dot{\gamma} - \Delta_\chi|| + ||\Delta_T||) / \lambda_{\text{min}}(Q).$$