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Stabilization of a Cascade System of ODE-PDE Subject to Boundary Control Matched Disturbance*

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***Abstract*— In this paper, we are concerned with the boundary feedback stabilization of a cascade system of ODE-PDE with the external disturbance flowing the control end, where the PDE is described by a heat equation with Dirichlet interconnection. We use the sliding mode control (SMC) to deal with the disturbance. By the SMC approach, the disturbance is supposed to be bounded only. The existence and uniqueness of the solution for the closed-loop via SMC are proved, and the monotonicity of the "reaching condition" is presented without the differentiation of the sliding mode function, for which it may not always exist for the weak solution of the closed-loop system.**

***Index Terms*— Boundary control, Disturbance rejection, Sliding mode control, Cascade systems, Backstepping.**

I. Introduction

The stabilization of partial differential equations (PDEs) subject to external disturbances has received a lot of attention in recent years as practically motivated. How to generalize the existing methods dealing with external disturbances in ordinary differential equations (ODEs) to the PDEs is quite challenging, but not impossible. The backstepping approach, which was originally developed for PDEs in the ideal operational environment [19], [20], has been applied to the stabilization of wave equations with harmonic disturbances in either boundary input or in observation [8], [9], [10], [11]. Due to its good performance in disturbance rejection and insensibility to uncertainties, the sliding mode control (SMC) has also been applied to some PDEs [3], [16], [17], [21]. Recently, the boundary SMC controllers are designed for one-dimensional heat, wave, Euler-Bernoulli, and Schrdinger equations with boundary input disturbance respectively in [2], [4], [5], [6], [18] and the active disturbance rejection control (ADRC) method has been successfully applied to the attenuation of disturbance for one-dimensional anti-stable wave equation in [11]. Another powerful method in dealing with disturbances is based on Lyapunov functional approach [12], [7], [13]. Only the PDEs are considered in the above works. To the best of our knowledge, there are few works that consider the stabilization of the cascade systems of ODE-PDE subject to external disturbances. Such systems have

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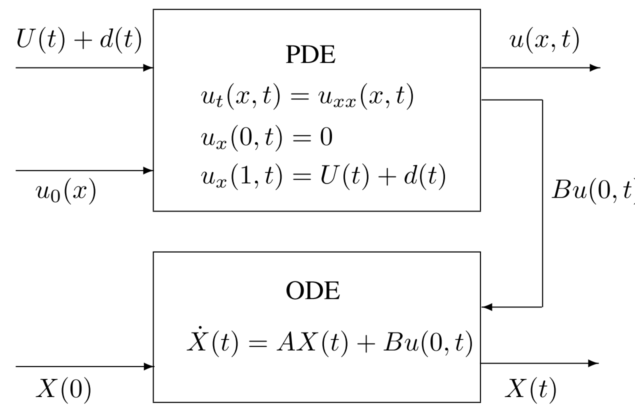

Fig. 1. Block diagram of a cascade system of ODE-PDE

taken place in many aspects such as electromagnetic coupling, mechanical coupling, and coupled chemical reactions.

In this paper, we consider the stabilization of a cascade system of ODE-PDE with Dirichlet interconnection :

$$\begin{cases} \dot X(t) & = AX(t)+Bu(0,t),\ t>0,\\ u_t(x,t) & = u_{xx}(x,t),\ x\in(0,1),\ t>0,\\ u_x(0,t) & = 0,\ u_x(1,t)=U(t)+d(t) \end{cases} \tag{1.1}$$

where $X(t)\in\mathbf{R}^{n\times 1}$ and $u(x,t)\in\mathbf{R}$ are the states of ODE and PDE respectively, $U(t)\in\mathbf{R}$ is the control input to the entire system, $A\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times 1}$, and $d(t)\in\mathbf{R}$ is the external disturbance at the control end. The unknown disturbance $d(t)$ is supposed to be bounded, i.e., $|d(t)|\le M$ for some $M>0$ and all $t\ge 0$.

The objective of the paper is to design a control $U(t)$ which can stabilize system (1.1) in $\mathbf{R}^n\times L^2(0,1)$ with the matched disturbance $d(t)$.

The rest of the paper is organized as follows. Section II is devoted to the design of sliding mode surface and the Lyapunov method is used to show the system in the sliding

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mode surface is exponentially stable. In Section III, the sliding mode control (SMC) is designed and the existence and uniqueness of solution of the closed-loop system with Dirichlet interconnection are proved. The finite time "reaching condition" is presented. Some concluding remarks are presented in Section IV.

II. Design of Sliding Mode Surface

We introduce a transformation $(X, u) \mapsto (X, w)$ in the form (see [14]):

$$\begin{aligned} X(t) &= X(t) \\ w(x,t) &= u(x,t) - \int_0^x q(x,y)u(y,t)dy \\ &\quad -\gamma(x)X(t) \end{aligned} \tag{2.2}$$

where

$$q(x,y) = \int_0^{x-y} \gamma(\sigma)Bd\sigma,$$

$$\gamma(x) = [K \quad 0]e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}x} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

where I denotes an $n\times n$ identify matrix. This transformation brings system (1.1) into the following system:

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw(0,t), \\ w_t(x,t) = w_{xx}(x,t), \\ w_x(0,t) = 0, \\ w_x(1,t) = U(t) + d(t) \\ \qquad - \int_0^1 q_x(1,y)u(y,t)dy - \gamma'(1)X(t), \end{cases} \tag{2.3}$$

where K is chosen such that $A+BK$ is Hurwitz and has real negative eigenvalues. The transformation (2.2) is invertible, that is

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy + \psi(x)X(t)$$

where

$$l(x,y) = \int_0^{x-y} \psi(\xi)Bd\xi,$$

$$\psi(x) = [K \quad 0]e^{\begin{bmatrix} 0 & A+BK \\ I & 0 \end{bmatrix}x} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Next we introduce the following transformation $(X, w) \mapsto (X, z)$:

$$\begin{aligned} X(t) &= X(t) \\ z(x,t) &= w(x,t) - \int_0^x k(x,y)w(y,t)dy \end{aligned} \tag{2.4}$$

where

$$k(x,y) = -cx\frac{I_1(\sqrt{c(x^2-y^2)})}{\sqrt{c(x^2-y^2)}}, c > 0, 0 \le y \le x \le 1 \tag{2.5}$$

and I_1 is the modified Bessel function. We obtain the target system

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bz(0,t), \\ z_t(x,t) = z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = U(t) + d(t) - \int_0^1 q_x(1,y)u(y,t)dy \\ \qquad -\gamma'(1)X(t) - k(1,1)w(1,t) \\ \qquad - \int_0^1 k_x(1,y)w(y,t)dy. \end{cases} \tag{2.6}$$

The inverse of the transformation (2.4) can be found as follows

$$w(x,t) = z(x,t) + \int_0^x p(x,y)z(y,t)dy \tag{2.7}$$

where

$$p(x,y) = -cx\frac{J_1(\sqrt{c(x^2-y^2)})}{\sqrt{c(x^2-y^2)}}, \quad 0 \le y \le x \le 1,$$

and J_1 is a Bessel function.

Let us consider system (1.1) and (2.6) in the state space $\mathcal{H} = \mathbf{R}^n \times L^2(0,1)$ with inner product given by

$$\begin{aligned} \langle (X,f), \ (Y,g) \rangle &= X^\top Y + \int_0^1 f(x)g(x)dx, \\ &\quad \forall (X,f),(Y,g) \in \mathcal{H}. \end{aligned} \tag{2.8}$$

Design the sliding mode surface $\mathcal{S}_z$ for system (2.6) as a closed-subspace of $\mathcal{H}$:

$$\mathcal{S}_z = \left\{ (X,f) \in \mathcal{H} \ \Big| \ \int_0^1 f(x)dx = 0 \right\}. \tag{2.9}$$

It is obvious that $\mathcal{S}_z$ is an infinite-dimensional manifold of $\mathcal{H}$, on which the system (2.6) becomes

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bz(0,t), \\ z_t(x,t) = z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \ \int_0^1 z(x,t)dx = 0. \end{cases} \tag{2.10}$$

Theorem 2.1: The PDE-ODE system (2.10) in the sliding mode surface $\mathcal{S}_z$ is exponentially stable in $\mathbf{R}^n \times L^2(0,1)$, that is, there are positive constants C_1, C_2 such that

$$|X(t)|^2 + \|z(t)\|^2 \le C_1 e^{-C_2 t}(|X_0|^2 + \|z_0\|^2) \tag{2.11}$$

where $|\cdot|$ and $\|\cdot\|$ denote the norms in $\mathbf{R}^n$ and $L^2(0,1)$ respectively.

Proof: We consider a Lyapunov functional $V(t) = X(t)^\top PX(t) + \frac{a}{2}\|z\|^2$, where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^T P = -Q$$

for some $Q = Q^T > 0$, and the parameter $a > 0$ is the design parameter. By integrating from 0 to 1 in x for the two sides of the second equation of system (2.10), it follows from $\int_0^1 z(x,t)dx = 0$ that $z_x(1,t) = 0$. Let $\lambda_{\min}(Q)$ be the minimum eigenvalue of matrix Q. By using Agmon's inequality ([15]), we have $z^2(0,t) \leq \frac{5}{2}\|z(x,t)\|^2 + 3\|z_x(x,t)\|^2$. Now by taking a derivative of the Lyapunov function along the solutions of system (2.10), we get

$$\begin{aligned}
\dot{V}(t) &= -X^\top QX + 2X^\top PBz(0,t) - a\|z_x(x,t)\|^2 \\
&\quad -ca\|z(x,t)\|^2 \\
&\leq -\frac{\lambda_{\min}(Q)}{2}|X|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)}z^2(0,t) \\
&\quad -a\|z_x(x,t)\|^2 - ca\|z(x,t)\|^2 \\
&\leq -\frac{\lambda_{\min}(Q)}{2}|X|^2 - \left(ca - \frac{5|PB|^2}{\lambda_{\min}(Q)}\right)\|z(x,t)\|^2 \\
&\quad -\left(a - \frac{6|PB|^2}{\lambda_{\min}(Q)}\right)\|z_x(x,t)\|^2. \qquad (2.12)
\end{aligned}$$

Let

$$a > \frac{6|PB|^2}{\lambda_{\min}(Q)}, \quad ca > \frac{5|PB|^2}{\lambda_{\min}(Q)}.$$

Then we get $\dot{V}(t) \leq -C_2 V$, where

$$C_2 = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{c}{2} - \frac{10|PB|^2}{a\lambda_{\min}(Q)}\right\} > 0.$$

Hence, we can obtain (2.11). ■

Transforming $\mathcal{S}_z$ by (2.4) into the original system (1.1), that is,

$$\begin{aligned}
S_z(t) &= \int_0^1 z(x,t)dx = \int_0^1 w(x,t)dx \\
&\quad - \int_0^1\int_0^x k(x,y)w(y,t)dydx \\
&= \int_0^1 u(x,t)dx - \int_0^1\int_0^x q(x,y)u(y,t)dydx \\
&\quad - \int_0^1 \gamma(x)X(t)dx - \int_0^1\int_0^x k(x,y)u(y,t)dydx \\
&\quad - \int_0^1\int_0^x k(x,y)\int_0^y q(y,\tau)u(\tau,t)d\tau dydx \\
&\quad - \int_0^1\int_0^x k(x,y)\gamma(y)X(t)dydx = 0 \qquad (2.13)
\end{aligned}$$

we have the following proposition.

Proposition 2.1: The original system (1.1) in the sliding mode surface (2.13) becomes

$$\begin{cases}
\dot{X}(t) = AX(t) + Bu(0,t), \\
u_t(x,t) = u_{xx}(x,t), \\
u_x(0,t) = 0, \\
\int_0^1 u(x,t)dx = \int_0^1\int_0^x q(x,y)u(y,t)dydx \\
\quad + \int_0^1 \gamma(x)X(t)dx + \int_0^1\int_0^x k(x,y)u(y,t)dydx \\
\quad + \int_0^1\int_0^x k(x,y)\int_0^y q(y,\tau)u(\tau,t)d\tau dydx \\
\quad + \int_0^1\int_0^x k(x,y)\gamma(y)X(t)dydx
\end{cases} \qquad (2.14)$$

which is exponentially stable in $\mathbf{R}^n \times L^2(0,1)$, by Theorem 2.1 and the equivalence between (2.10) and (2.14).

III. State Feedback Controller

To facilitate the control design, we differentiate (2.13) with respect to t to obtain

$$\begin{aligned}
\dot{S}_z(t) &= \int_0^1 z_t(x,t)dx \\
&= \int_0^1 [z_{xx}(x,t) - cz(x,t)]dx \\
&= z_x(1,t) - cS_z(t)
\end{aligned}$$

and hence

$$\begin{aligned}
\dot{S}_z(t) &= U(t) + d(t) - \int_0^1 q_x(1,y)u(y,t)dy \\
&\quad -\gamma'(1)X(t) - k(1,1)w(1,t) \\
&\quad -\int_0^1 k_x(1,y)w(y,t)dy - cS_z(t).
\end{aligned}$$

Design the feedback controller:

$$\begin{aligned}
U(t) &= U_0(t) + \int_0^1 q_x(1,y)u(y,t)dy + k(1,1)w(1,t) \\
&\quad +\gamma'(1)X(t) + \int_0^1 k_x(1,y)w(y,t)dy \qquad (3.15)
\end{aligned}$$

where U_0 is a new control. Then we have

$$\dot{S}_z(t) = U_0(t) + d(t) - cS_z(t).$$

Let $U_0(t) = -(M+\eta)\text{sign}(S_z(t))$. Then we get

$$\dot{S}_z(t) = -(M+\eta)\text{sign}(S_z(t)) + d(t) - cS_z(t) \qquad (3.16)$$

and

$$\begin{aligned}
S_z(t)\dot{S}_z(t) &= -(M+\eta)\text{sign}(S_z(t))S_z(t) + d(t)S_z(t) \\
&\quad -c|S_z(t)|^2 \leq -\eta|S_z(t)|, \quad \eta > 0. \qquad (3.17)
\end{aligned}$$

Note that (3.17) is just the well-known "reaching condition" for system (2.6). The sliding mode controller is

$$\begin{aligned} U(t) &= \int_0^1 q_x(1,y)u(y,t)dy + \gamma'(1)X(t) \qquad (3.18)\\ &\quad +k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)dy \\ &\quad -(M+\eta)\mathrm{sign}(S_z(t)) \end{aligned}$$

Remark 3.1: Due to the use of sign function $\mathrm{sign}(S_z(t))$, the control signal $U(t)$ (3.18) becomes discontinuous, which may excite unmodeled high-frequency plant dynamics and cause the chattering phenomenon. To avoid the undesired chattering phenomenon, we can replace the sign function in $U(t)$ with the following saturation function in the simulation:

$$\mathrm{sat}(*) = \begin{cases} 1, & \text{if } * > \epsilon \\ \frac{*}{\epsilon}, & \text{if } |*| \le \epsilon \\ -1, & \text{if } * < -\epsilon \end{cases}$$

where ϵ is a small positive constant.

The closed-loop system of system (2.6) under the state feedback controller (3.18) is

$$\begin{cases} \dot{X}(t) &= (A+BK)X(t) + Bz(0,t), \\ z_t(x,t) &= z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) &= 0, \\ z_x(1,t) &= -(M+\eta)\mathrm{sign}(S_z(t)) + d(t) = \tilde{d}(t). \end{cases} \qquad (3.19)$$

Now we are in position to consider the existence and uniqueness of the solution to (3.19) and the finite time "reaching condition" to the sliding mode surface S_z.

Write system (3.19) as

$$\frac{d}{dt}Z(\cdot,t) = \mathcal{A}Z(\cdot,t) + \mathcal{B}\tilde{d}(t), \qquad (3.20)$$

where $Z(t) = (X(t),\ z(\cdot,t))$, $\mathcal{B} = (0,\ \delta(x-1))$, and $\mathcal{A}$ is a linear operator defined in $\mathbf{R}^n \times L^2(0,1)$ by

$$\begin{aligned} &\mathcal{A}(X,f) = ((A+BK)X + Bf(0), f'' - cf), \\ &D(\mathcal{A}) = \{(X,f) \in \mathbf{R}^n \times H^2(0,1) | f'(0) = f'(1) = 0\}. \end{aligned} \qquad (3.21)$$

Moreover, we have $\mathcal{A}^*$, the adjoint operator of $\mathcal{A}$,

$$\begin{cases} \mathcal{A}^*(Y,g) = ((A+BK)^\top Y, g'' - cg), \forall (Y,g) \in D(\mathcal{A}^*) \\ D(\mathcal{A}^*) = \left\{ (Y,g) \in \mathbf{R}^n \times H^2(0,1) \middle| \begin{array}{l} g'(0) = -B^\top Y \\ g'(1) = 0 \end{array} \right. \end{cases} \qquad (3.22)$$

and the dual system of (3.20) is given by

$$\begin{cases} \dot{X}^*(t) = (A+BK)^\top X^*(t), \\ z_t^*(x,t) = z_{xx}^*(x,t) - cz^*(x,t), \\ z_x^*(0,t) = -B^\top X^*,\ z_x^*(1,t) = 0, \\ y(t) = \mathcal{B}^* \begin{pmatrix} X^* \\ z^*(x,t) \end{pmatrix} = z^*(1,t). \end{cases} \qquad (3.23)$$

Theorem 3.1: Suppose that d is measurable and $|d(t)| \le M$ for all $t \ge 0$, and S_z be defined by (2.9). Let $\mathcal{A}^*$ be given by (3.22), let $\lambda_j^0,\quad j = 1,2,\ldots,n$ be the simple real and negative eigenvalue of $(A+BK)^\top$ with the corresponding eigenvector X_{0j}^*, and assume that

$$\lambda_j^0 \notin \left\{ \lambda_m^1 = -c - (m\pi)^2,\ m = 0,1,2,\ldots \right\}.$$

Then we have the eigenvalues of $\mathcal{A}^*$ given by

$$\{\lambda_j^0,\quad j = 1,2,\ldots,n\} \cup \{\lambda_m^1,\ m = 0,1,2,\ldots\} \qquad (3.24)$$

and the corresponding eigenfunctions with respect to λ_j^0 and λ_m^1 given by respectively

$$\begin{aligned} &Z_j^0(x) = \left(X_{0j}^*, z_{0j}^*(x)\right),\ Z_m^1(x) = (0, z_{1m}^*(x)), \\ &j = 1,2,\ldots,n, m = 0,1,2,\ldots \end{aligned} \qquad (3.25)$$

where

$$\begin{cases} z_{0j}^*(x) = \dfrac{B^\top X_{0j}^* \cosh \sqrt{\lambda_j^0 + c}(1-x)}{\sqrt{\lambda_j^0 + c}\sinh\sqrt{\lambda_j^0 + c}}, \\ z_{1m}^*(x) = \cos m\pi x. \end{cases} \qquad (3.26)$$

Moreover $\{Z_j^0(x), Z_m^1(x), j = 1,2,\cdots,n, m = 0,1,\cdots\}$ forms a Riesz basis for $\mathbf{R}^n \times L^2(0,1)$. Therefore for any $(X(0), z(\cdot,0)) \in \mathcal{H}$ and $S_z(0) \ne 0$, there exists a $t_{max} > 0$ such that (3.19) admits a unique solution $(X,z) \in C(0,t_{max};\mathcal{H})$ and $S_z(t) = 0$ for all $t \ge t_{max}$.

Returning back to the system (1.1) under the transformation (2.2) and (2.4), feedback control (3.18), we obtain the main result of this section.

Theorem 3.2: Suppose that $|d(t)| \le M$ for all $t \ge 0$, and let S_U be the sliding mode function given by

$$S_U(t) = \int_0^1 w(x,t)dx - \int_0^1 \int_0^x k(x,y)w(y,t)dydx. \qquad (3.27)$$

where $w(x,t)$ is given by (2.2). Then for any $(X(0), u(\cdot,0)) \in \mathcal{H}$, $S_U(0) \ne 0$, there exists a $t_{max} > 0$ such that the closed-loop system of (1.1) under the feedback control (3.18) is

$$\begin{cases} \dot{X}(t) &= AX(t) + Bu(0,t), \\ u_t(x,t) &= u_{xx}(x,t), \\ u_x(0,t) &= 0, \\ u_x(1,t) &= \displaystyle\int_0^x q_x(x,y)u(y,t)dy + \gamma'(1)X(t) \\ &\quad +k(1,1)w(1,t) + \displaystyle\int_0^1 k_x(1,y)w(y,t)dy \\ &\quad -(M+\eta)\mathrm{sign}(S_z(t)) + d(t) \end{cases} \qquad (3.28)$$

which admits a unique solution $(X,u) \in C(0,t_{max};\mathcal{H})$ and $S_U(t) = 0$ for all $t \ge t_{max}$. On the sliding mode surface

$S_U(t)=0$, the system (1.1) becomes

$$\begin{cases} \dot{X}(t)=AX(t)+Bu(0,t), \\ u_t(x,t)=u_{xx}(x,t), \\ u_x(0,t)=0, \\ \displaystyle\int_0^1 w(x,t)dx-\int_0^1\int_0^x k(x,y)w(y,t)dydx=0 \end{cases} \tag{3.29}$$

which is equivalent to (2.10) and hence is exponentially stable in $\mathcal{H}$ with the decay rate $-c$.

It is remarked that system (3.19) is equivalent to system (3.28) under the equivalent transformation (2.2) and (2.4).

IV. Conclusion

In this paper, the sliding mode control (SMC) is designed to achieve the stabilization of a cascade system of ODE-PDE subject to boundary control matched disturbances. The existence and uniqueness of the solution for the closed-loop via SMC have been proved. The effectiveness of the proposed method is validated with numerical simulations.

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