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Adaptive NN Control of Strict-feedback Systems Using ISS-modular Approach

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***Abstract*— In this paper, adaptive neural network control is investigated for a general class of strict-feedback systems using "ISS-modular" approach. The closed-loop system consists of two interconnected subsystems: the state error subsystem and the weight estimation subsystem. First, a neural controller is designed to achieve ISS for the state error subsystem with respect to the neural weight estimation errors. Then, a neural weight estimator is designed to achieve ISS for the weight estimation subsystem with respect to the system state errors. Finally, the stability of the entire closed-loop system is guaranteed by the small-gain theorem. The "ISS-modular" approach avoids the construction of an overall Lyapunov function for the closed-loop system, and overcomes the controller singularity problem completely. The simulation studies demonstrate the effectiveness of the proposed control method.**

I. Introduction

In recent years, adaptive control of nonlinear systems with parametric and functional uncertainties has received a great deal of attention in the nonlinear control community. In particular, the appearance of recursive backstepping design makes a significant development for the adaptive control of some classes of nonlinear systems with the triangular structure form [1] [2]. In the literature, there are many works which study the class of strict-feedback systems, which is given in a general form as [1]

$$\begin{aligned} \dot{x}_i &= g_i(\bar{x}_i)x_{i+1} + f_i(\bar{x}_i),\ 1 \le i \le n-1 \\ \dot{x}_n &= g_n(\bar{x}_n)u + f_n(\bar{x}_n),\ n \ge 2 \\ y &= x_1 \end{aligned} \tag{1}$$

where $\bar{x}_i = [x_1, \cdots, x_i]^T \in R^i$, $i = 1, \cdots, n$, $u \in R$, $y \in R$ are state variables, system input and output, respectively; $f_i(\cdot)$ and $g_i(\cdot)$, $i = 1, \cdots, n$ are unknown smooth functions.

For the control of the uncertain strict-feedback system (1), adaptive control design has been extensively studied using the celebrated systematic feedback linearization technique. If $f_i(\cdot)$ and $g_i(\cdot)$, $i = 1, \cdots, n$ are known smooth functions, the feedback linearization type controllers $\alpha_i = \frac{1}{g_i(\cdot)}(-f_i(\cdot) + v_i)$, where v_i are new control variables for the resulting linear control system, can achieve the control objective effectively [1]. To deal with unknown functional uncertainties, neural networks (NNs) are often used as on-line approximators, owing to their universal approximation capabilities, learning and adaptation, parallel distributed structures [3] [4] [5] and [6]. When only $f_i(\cdot)$ are unknown smooth functions, we can use neural networks approximations $\hat{f}_i(\cdot)$ to compensate for the uncertainties $f_i(\cdot)$, which results in the modified feedback linearization type controllers $\alpha_i = \frac{1}{g_i(\cdot)}(-\hat{f}_i(\cdot) + v_i)$ [7]. When both $f_i(\cdot)$ and $g_i(\cdot)$ are unknown, the problem becomes complex. In this case, the indirect feedback linearization type controllers $\alpha_i = \frac{1}{\hat{g}_i(\cdot)}(-\hat{f}_i(\cdot) + v_i)$ could be considered, where $\hat{f}_i(\cdot)$ and $\hat{g}_i(\cdot)$ are the estimates of $f_i(\cdot)$ and $g_i(\cdot)$, respectively. When $\hat{g}_i(\cdot) \to 0$, the controller singularity problem arises. There are many works dealing with the controller singularity problem in the control of the uncertain strict-feedback system (1).

Special precautions have to be taken to avoid the possibility of control singularity because the standard adaptive law does not guarantee the estimate of the control gain be kept away from zero. To deal with such a problem, projection algorithms and dead zone methods have been frequently applied to avoid the possible controller singularity problem. The project algorithm seems very simple, but it relies on the exact knowledge of the ranges of the parameters. Such a requirement limits its usefulness, e.g., the lack of physical meanings in neural network weights makes the construction of a meaningful project extremely difficult in adaptive neural network control design, if not impossible. On the other hand, the dead zone method is only a practical remedy, and the problem remained open for a more elegant mathematical solution with the beauty of mathematical rigor until the pioneering introduction of the novel family of integral Lyapunov functions in a series of works [3] [8] [9] and [10]. In [10], through the definition of a novel integral-type Lyapunov function, direct adaptive neural control was proposed for the first time for strict-feedback system (1), where $f_i(\cdot)$ and $g_i(\cdot)$, $i = 1, \cdots, n$ are all unknown smooth functions. In the approach, a desired feedback control law is first proved to be in existence, then neural networks are used to parameterize the desired feedback control law, finally adaptive techniques are used to tune the weights of neural networks for closed-loop stability. The possible controller singularity problem usually met in adaptive control is completely overcome with mathematic rigor. In [11], direct adaptive NN control is presented for nonlinear strict-feedback system (1), where $f_i(\cdot)$ and $g_i(\cdot)$, $i = 1, \cdots, n$ are unknown smooth functions, and moreover, the affine term $g_n(\cdot)$ is assumed to be independent of state x_n. By exploiting this property, the developed neural control scheme avoids the controller singularity problem completely, and stability of the resulting adaptive system is guaranteed without the requirement for integral-type Lyapunov functions.

In this paper, motivated by the work [12], we present

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an ISS-modular approach for strict-feedback system (1) by employing the input-to-state stability (ISS) analysis [13] [14] and the small gain theorem [15]. The proposed adaptive neural control approach will achieve "ISS-modularity" of the controller-estimator pair, i.e., a neural controller is designed to achieve ISS for the state error subsystem with respect to the neural weight estimation errors, and a neural weight estimator is designed to achieve ISS for the weight estimation subsystem with respect to the system state errors. The stability of the entire closed-loop system is guaranteed by the small-gain theorem. The ISS-modular approach provides a simple and effective way for adaptive neural control of system (1), which can avoid the construction of an overall Lyapunov function for the entire system, and can avoid the controller singularity problem completely without complicate integration operation [10] and restrictive assumptions like [11].

The paper is organized as follows. For completeness, Section II presents some preliminary knowledge, include some notations, assumptions and radial basis function neural network (RBFNN), which are used in the later adaptive neural control design. In Section III, ISS-modular approach is investigated for adaptive neural control of strict-feedback system. Simulation studies are included in Section IV to demonstrate the effectiveness of the proposed approach. Conclusion is finally followed in Section V.

II. Problem Formulation and Preliminaries

A. Problem Formulation and Notations

Definition 1: A continuous function $\alpha : R_+ \to R_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : R_+ \times R_+ \to R_+$ is said to belong to class $\mathcal{KL}$, if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

The control objective is to design a direct adaptive neural controller for system (1) such that (i) all the signals in the closed-loop system remain uniformly ultimately bounded, and (ii) the output y follows a desired trajectory y_d generated from the following smooth, bounded reference model:

$$\begin{aligned} \dot{x}_{di} &= f_{di}(x_d),\ 1 \le i \le n, \\ y_d &= x_{d1}, \end{aligned} \tag{2}$$

where $x_d = [x_{d1}, x_{d2}, ..., x_{dn}]^T \in R^n$ are the states, $y_d \in R$ is the system output, $f_{di}(\cdot), i = 1, 2, ..., n$ are known smooth nonlinear functions. Assume that the states of the reference model remain bounded, i.e., $x_d \in \Omega_d, \forall t \ge 0$.

Assumption 1: The signs of $g_i(\cdot)$ are known, and there exist constants $\bar{g}_i \ge \underline{g}_i > 0$ such that $\bar{g}_i \ge |g_i(\cdot)| \ge \underline{g}_i$, $\forall \bar{x}_n \in \Omega \subset R^n$.

The above assumption implies that smooth functions $g_i(\cdot)$ are strictly either positive or negative. Without losing generality, we shall assume $\bar{g}_i \ge g_i(\cdot) \ge \underline{g}_i > 0$, $\forall \bar{x}_n \in \Omega \subset R^n$.

B. Radial Basis Function Neural Network

In this paper, the following RBFNN [16] is used to approximate the continuous function $h(Z) : R^q \to R$,

$$h_{nn}(Z) = W^T S(Z) \tag{3}$$

where the input vector $Z \in \Omega \subset R^q$, weight vector $W = [w_1, w_2, ..., w_l]^T \in R^l$, the NN node number $l > 1$; and $S(Z) = [s_1(Z), ..., s_l(Z)]^T$, with $s_i(Z)$ being chosen as the commonly used Gaussian functions, which have the form

$$s_i(Z) = \exp\Big[\frac{-(Z - \mu_i)^T (Z - \mu_i)}{\eta^2}\Big],\ i = 1, 2, ..., l \tag{4}$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, ..., \mu_{iq}]^T$ is the center of the receptive field and η is the width of the Gaussian function.

It has been proven that network (3) can approximate any continuous function over a compact set $\Omega_Z \subset R^q$ to arbitrary any accuracy as

$$h(Z) = W^{*T} S(Z) + \varepsilon(Z),\ \forall Z \in \Omega_Z \tag{5}$$

where W^* is ideal constant weights, and $\varepsilon(Z)$ is the approximation error ($\varepsilon(Z)$ is denoted as ε to simplify the notation).

Assumption 2: There exist ideal constant weights W^* such that $|\varepsilon| \le \varepsilon^*$ with constant $\varepsilon^* > 0$ for all $Z \in \Omega_Z$. Moreover, W^* is bounded by $\|W^*\| \le \bar{W}^*$ on the compact set Ω_Z.

It is clear that W^* is usually unknown and need to be estimated in function approximation. Let $\hat{W}$ be the estimates of W^*, and the weight estimation error be $\tilde{W} = \hat{W} - W^*$.

For Gaussian RBF networks, the following lemma provides an upper bound on the 2-norm of vector $S(Z)$, which is essential in proving our main result.

Lemma 1: [17] Consider the Gaussian RBF networks (3)(4). Let $\rho := \frac{1}{2} \min_{i \ne j} \|\mu_i - \mu_j\|$, and let q be the dimension of input Z, and η be the width of Gaussian function (as in (4)). Then we may take an upper bound of $\|S(Z)\|$ as

$$\|S(Z)\| \le s^* \tag{6}$$

where $s^* = \sum_{k=0}^{\infty} 3q(k + 2)^{q-1} e^{-2\rho^2 k^2/\eta^2}$.

Remark 1: It can be easily proven that the sum $\sum_{k=0}^{\infty} 3q(k + 2)^{q-1} e^{-2\rho^2 k^2/\eta^2}$ has a limited value s^*, since the infinite series $\{3q(k + 2)^{q-1} e^{-2\rho^2 k^2/\eta^2}\}(k = 0, ..., \infty)$ is convergent by the Ratio Test Theorem [18]. Note also that this limited value s^* is independent of Z (the NN inputs) and l (the dimension of neural weights W).

III. Control Design

In this section, we investigate adaptive NN control design using ISS-modular approach, which includes three steps: First, to design an adaptive NN controller such that the system state errors are ISS with respect to the NN weight estimation errors; second, to design a neural weight estimator such that the NN weight estimation errors are ISS with respect to the system state errors; and finally, the stability of the entire closed-loop system will be guaranteed by using the small-gain theorem.

A. ISS for System State Errors

In this subsection, direct adaptive NN design is combined with backstepping design. At each recursive step $i(1 \leq i \leq n)$ in backstepping design, the desired virtual control α_i^* and the desired control $u^* = \alpha_n^*$ are firstly shown to exist which possess some desired stabilizing properties. The desired virtual control $\alpha_i^*(1 \leq i \leq n)$ contains uncertainties $f_i(\cdot)$ and $g_i(\cdot)(1 \leq i \leq n)$, and thus cannot be implemented in practice. Then, the virtual control α_i and the practical control u are constructed by using RBF neural networks $W_i^T S_i(Z_i)$ to parameterize the unknown parts in the desired virtual control α_i^* and the desired control u^*. By designing the stabilizing functions $\alpha_i(1 \leq i \leq n)$, the state error subsystem is obtained. The detailed design procedure is described in the following steps.

Step 1: Define $z_1 = x_1 - x_{d1}$. Its derivative is

$$\dot{z}_1 = f_1(x_1) + g_1(x_1)x_2 - \dot{x}_{d1}. \tag{7}$$

By viewing x_2 as a virtual control input, there exists a smooth desired control input α_1^* as

$$\alpha_1^* = -c_1 z_1 - \frac{1}{g_1}[f_1 - \dot{x}_{d1}] \tag{8}$$

where $c_1 > 0$ is a design constant. Since $f_1(x_1)$ and $g_1(x_1)$ are unknown smooth functions, the desired feedback control α_1^* cannot be implemented in practice. From (8), it can be seen that the unknown part $(1/g_1(x_1))(f_1(x_1) - \dot{x}_{d1})$ is a smooth function of x_1 and $\dot{x}_{d1}$. Denote

$$h_1(Z_1) = \frac{1}{g_1(x_1)}(f_1(x_1) - \dot{x}_{d1}) \tag{9}$$

where $Z_1 = [x_1, \dot{x}_{d1}]^T \subset R^2$. By employing an RBF neural network $W_1^T S_1(Z_1)$ to approximate $h_1(Z_1)$, α_1^* can be expressed as

$$\alpha_1^* = -c_1 z_1 - W_1^{*T} S_1(Z_1) - \varepsilon_1 \tag{10}$$

where W_1^* denotes the ideal constant weights, and $|\varepsilon_1| \leq \varepsilon_1^*$ is the approximation error with constant $\varepsilon_1^* > 0$. Since W_1^* is unknown, let $\hat{W}_1$ be the estimate of W_1^*.

Since x_2 is only taken as a virtual control, not as the real control input for the z_1-subsystem, by introducing the error variable $z_2 = x_2 - \alpha_1$ and choosing the virtual control

$$\alpha_1 = -c_1 z_1 - \hat{W}_1^T S_1(Z_1) \tag{11}$$

then we have

$$\begin{aligned} \dot{z}_1 &= f_1(x_1) + g_1(x_1)(z_2 + \alpha_1) - \dot{x}_{d1} \\ &= g_1(x_1)(z_2 - c_1 z_1 - \tilde{W}_1^T S_1(Z_1) + \varepsilon_1) \end{aligned} \tag{12}$$

where $\tilde{W}_1 = \hat{W}_1 - W_1^*$. Through out this paper, we shall define $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)^*$.

Step i $(2 \leq i \leq n-1)$**:** The derivative of $z_i = x_i - \alpha_{i-1}$ is

$$\dot{z}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} - \dot{\alpha}_{i-1} \tag{13}$$

By viewing x_{i+1} as a virtual control input to stabilize the $z_1, \cdots, z_i$-subsystem, there exists a desired feedback control

$$\alpha_i^* = x_{i+1} = -z_{i+1} - c_i z_i - \frac{1}{g_i(\bar{x}_i)}(f_i(\bar{x}_i) - \dot{\alpha}_{i-1}) \tag{14}$$

where

$$\dot{\alpha}_{i-1} = \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}(g_k(\bar{x}_k)x_{k+1} + f_k(\bar{x}_k)) + \phi_{i-1} \tag{15}$$

with $\phi_{i-1} = \partial\alpha_{i-1}/\partial x_d \dot{x}_d + \sum_{k=1}^{i-1}(\partial\alpha_{i-1}/\partial\hat{W}_k)\dot{\hat{W}}_k$ computable.

Let

$$h_i(Z_i) = \frac{1}{g_i(\bar{x}_i)}(f_i(\bar{x}_i) - \dot{\alpha}_{i-1}) \tag{16}$$

denote the unknown part in α_i^* (14), with

$$Z_i = [\bar{x}_i^T, \frac{\partial \alpha_{i-1}}{\partial x_1}, \cdots, \frac{\partial \alpha_{i-1}}{\partial x_{i-1}}, \phi_{i-1}]^T \subset R^{2i} \tag{17}$$

By employing an RBF neural network $W_i^T S_i(Z_i)$ to approximate $h_i(Z_i)$, α_i^* can be expressed as

$$\alpha_i^* = -z_{i+1} - c_i z_i - W_i^{*T} S_i(Z_i) - \varepsilon_i \tag{18}$$

Define the error variable $z_{i+1} = x_{i+1} - \alpha_i$ and choose the virtual control

$$\alpha_i = -z_{i+1} - c_i z_i - \hat{W}_i^T S_i(Z_i) \tag{19}$$

Then, we have

$$\begin{aligned} \dot{z}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)(z_{i+1} + \alpha_i) - \dot{\alpha}_{i-1} \\ &= g_i[z_{i+1} - z_{i-1} - c_i z_i - \tilde{W}_i^T S_i(Z_i) + \varepsilon_i] \end{aligned} \tag{20}$$

Step n: This is the final step. The derivative of $z_n = x_n - \alpha_{n-1}$ is

$$\dot{z}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u - \dot{\alpha}_{n-1} \tag{21}$$

There exists a desired feedback control

$$u^* = -z_{n-1} - c_n z_n - \frac{1}{g_n(\bar{x}_n)}(f_n(\bar{x}_n) - \dot{\alpha}_{n-1}) \tag{22}$$

where

$$\dot{\alpha}_{n-1} = \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}(g_k(\bar{x}_k)x_{k+1} + f_k(\bar{x}_k)) + \phi_{n-1} \tag{23}$$

with $\phi_{n-1} = \partial\alpha_{n-1}/\partial x_d \dot{x}_d + \sum_{k=1}^{n-1}(\partial\alpha_{i-1}/\partial\hat{W}_k)\dot{\hat{W}}_k$ computable.

Let

$$h_n(Z_n) = \frac{1}{g_n(\bar{x}_n)}(f_n(\bar{x}_n) - \dot{\alpha}_{n-1}) \tag{24}$$

denote the unknown part in u^* (22), with

$$Z_n = [\bar{x}_n^T, \frac{\partial \alpha_{n-1}}{\partial x_1}, \cdots, \frac{\partial \alpha_{n-1}}{\partial x_{n-1}}, \phi_{n-1}]^T \subset R^{2n} \tag{25}$$

By employing an RBF neural network $W_n^T S_n(Z_n)$ to approximate $h_n(Z_n)$, u^* can be expressed as

$$u^* = -z_{n-1} - c_n z_n - W_n^{*T} S_n(Z_n) - \varepsilon_n \tag{26}$$

Choose the practical control law as

$$u = -z_{n-1} - c_n z_n - \hat{W}_n^T S_n(Z_n) \tag{27}$$

Then, we have

$$\begin{aligned}\dot{z}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n)u - \dot{\alpha}_{n-1} \\ &= g_n(\bar{x}_n)[-z_{n-1} - c_n z_n - \tilde{W}_n^T S_n(Z_n) + \varepsilon_n]\end{aligned} \tag{28}$$

As such, we can obtain the state error subsystem as following by combining Eqs. (12), (20) and (28)

$$\begin{aligned}\dot{z}_1 &= g_1(x_1)(z_2 - c_1 z_1 - \tilde{W}_1^T S_1(Z_1) + \varepsilon_1) \\ \dot{z}_i &= g_i[z_{i+1} - z_{i-1} - c_i z_i - \tilde{W}_i^T S_i(Z_i) + \varepsilon_i] \\ \dot{z}_n &= g_n(\bar{x}_n)[-z_{n-1} - c_n z_n - \tilde{W}_n^T S_n(Z_n) + \varepsilon_n]\end{aligned} \tag{29}$$

where $i = 2, ..., n-1$.

Lemma 2: The state error subsystem (29), which can be viewed as a system with states $z = [z_1, \cdots, z_n]^T$, and inputs $\tilde{W} = [\tilde{W}_1^T, \cdots, \tilde{W}_n^T]^T$ and $\varepsilon = [\varepsilon_1, \cdots, \varepsilon_n]^T$, is input-to-state stable.

Proof: Consider the following quadratic function

$$V_z = \frac{1}{2}\sum_{i=1}^{n} z_i^2 \tag{30}$$

Its derivative along (29) is

$$\begin{aligned}\dot{V}_z &= -\sum_{i=1}^{n} c_i g_i z_i^2 + \sum_{i=1}^{n-1}(g_i - g_{i+1}) z_i z_{i+1} \\ &\quad -\sum_{i=1}^{n} g_i \tilde{W}_i^T S_i(Z_i) z_i + \sum_{i=1}^{n} g_i z_i \varepsilon_i \\ &\le -\sum_{i=1}^{n} c_i g_i z_i^2 + \sum_{i=1}^{n-1} |g_i - g_{i+1}| \frac{z_i^2 + z_{i+1}^2}{2} \\ &\quad +\sum_{i=1}^{n} g_i \|\tilde{W}_i\| \|S_i(Z_i)\| |z_i| + \sum_{i=1}^{n} g_i z_i \varepsilon_i \\ &\le -\sum_{i=1}^{n} c_{i0} g_i z_i^2 + \sum_{i=1}^{n-1} |g_i - g_{i+1}| \frac{z_i^2 + z_{i+1}^2}{2} \\ &\quad -\sum_{i=1}^{n} c_{i1} g_i z_i^2 + \sum_{i=1}^{n} g_i(-c_{i2} z_i^2 + s_i^* \|\tilde{W}_i\| |z_i|) \\ &\quad +\sum_{i=1}^{n} g_i(-c_{i3} z_i^2 + z_i \varepsilon_i) \\ &\le -\sum_{i=1}^{n} c_{i0} g_i z_i^2 + \frac{|g_1 - g_2|}{2} z_1^2 + \frac{|g_{n-1} - g_n|}{2} z_n^2 \\ &\quad +\sum_{i=2}^{n-1}\Big(\frac{|g_{i-1} - g_i|}{2} + \frac{|g_i - g_{i+1}|}{2}\Big) z_i^2 \\ &\quad -\sum_{i=1}^{n} c_{i1} g_i z_i^2 + \sum_{i=1}^{n} g_i(-c_{i2} z_i^2 + s_i^* \|\tilde{W}_i\| |z_i|) \\ &\quad +\sum_{i=1}^{n} g_i(-c_{i3} z_i^2 + z_i \varepsilon_i)\end{aligned} \tag{31}$$

where $c_i = \sum_{i=1,...n}^{j=0,...,3} c_{ij}$, with $c_{ij} > 0$. By completion of squares, the following inequalities hold:

$$\begin{aligned}-c_{i2} z_i^2 + s_i^* \|\tilde{W}_i\| |z_i| &\le \frac{s_i^{*2} \|\tilde{W}_i\|^2}{4c_{i2}} \\ -c_{i3} z_i^2 + z_i \varepsilon_i &\le \frac{\varepsilon_i^2}{4c_{i3}}\end{aligned} \tag{32}$$

and from Assumption 1, we have

$$|g_i - g_{i+1}| \le g_{i+1}^i \tag{33}$$

where $g_{i+1}^i = \max(\bar{g}_i, \bar{g}_{i+1}) - \min(\underline{g}_i, \underline{g}_{i+1})$. Choosing $c_{10} \ge g_2^1 / 2\underline{g}_1$, $c_{i0} \ge (g_i^{i-1} + g_{i+1}^i)/2\underline{g}_i$ $(i = 2, ..., n-1)$, $c_{n0} \ge g_n^{n-1}/2\underline{g}_n$, equation (31) becomes

$$\begin{aligned}\dot{V}_z &\le -\sum_{i=1}^{n} c_{i1} g_i z_i^2 + \sum_{i=1}^{n} \frac{g_i s_i^{*2} \|\tilde{W}_i\|^2}{4c_{i2}} + \sum_{i=1}^{n} \frac{g_i \varepsilon_i^2}{4c_{i3}} \\ &\le -\frac{c_{*1}\underline{g}_*}{2}\|z\|^2 - \Big(\frac{c_{*1}\underline{g}_*}{2}\|z\|^2 - \frac{s^{*2}\bar{g}_* \|\tilde{W}\|^2}{4c_{*2}} \\ &\quad -\frac{\bar{g}_* \|\varepsilon\|^2}{4c_{*3}}\Big)\end{aligned} \tag{34}$$

where $\underline{g}_* = \min_{1\le i\le n} \underline{g}_i$, $\bar{g}_* = \max_{1\le i\le n} \bar{g}_i$, $s^* = \max_{1\le i\le n} s_i^*$, $c_{*1} = \min_{1\le i\le n} c_{i1}$, $c_{*2} = \min_{1\le i\le n} c_{i2}$, and $c_{*3} = \min_{1\le i\le n} c_{i3}$.

As

$$\|z\| > \frac{s^*\sqrt{\bar{g}_*}}{\sqrt{2c_{*1}c_{*2}\underline{g}_*}}\|\tilde{W}\| + \frac{\sqrt{\bar{g}_*}}{\sqrt{2c_{*1}c_{*3}\underline{g}_*}}\|\varepsilon\| \tag{35}$$

implies $\dot{V}_z < -\alpha_z(\|z\|)$, where $\alpha_z(r) = (\frac{c_{*1}\underline{g}_*}{2})r^2$. Thus, according to [19], the state error subsystem (29) is input-to-state stable, with gain functions

$$\gamma_1^{\tilde{W}}(r) = \frac{s^*\sqrt{\bar{g}_*}}{\sqrt{2c_{*1}c_{*2}\underline{g}_*}} r, \quad \gamma_1^{\varepsilon}(r) = \frac{\sqrt{\bar{g}_*}}{\sqrt{2c_{*1}c_{*3}\underline{g}_*}} r. \qquad \diamondsuit$$

B. ISS for NN Weight Estimation Errors

In this subsection, we will make the neural weight estimation subsystem input-to-state stable with respect to the system state errors z. Consider the following Lyapunov-based $\sigma-$modification neural weight estimator [20] [21] [22]:

$$\dot{\tilde{W}}_i = \dot{\hat{W}}_i = \Gamma_i[S_i(Z_i)z_i - \sigma_i \hat{W}_i] \tag{36}$$

where $\Gamma_i = \Gamma_i^T > 0$, and $\sigma_i > 0$, $i = 1, ..., n$ are positive constant design parameters. Since $\hat{W}_i = \tilde{W}_i + W_i^*$, (36) can be rewritten as

$$\dot{\tilde{W}} = \Gamma[S(Z)z - \Theta\tilde{W} - \Theta W^*] \tag{37}$$

where $S(Z) = diag\{S_1(Z_1), ..., S_n(Z_n)\}$, $\Gamma = diag\{\Gamma_1, ..., \Gamma_n\}$, and $\Theta = diag\{\sigma_1 I, ..., \sigma_n I\}$.

Lemma 3: The neural weight estimation subsystem (37), which can be viewed as as system with state $\tilde{W}$, inputs z and W^* (the ideal NN weights), is input-to-state stable with respect to z and W^*.

Proof: Consider the following quadratic function

$$V_{\tilde{W}} = \frac{1}{2}\|\tilde{W}\|^2 \tag{38}$$

Its derivative along the trajectories of (37) is

$$\begin{aligned} \dot{V}_{\tilde{W}} &= \tilde{W}^T \dot{\tilde{W}} \\ &= \tilde{W}^T \Gamma[S(Z)z - \Theta\tilde{W} - \Theta W^*] \\ &= \tilde{W}^T \Gamma[-(1-\eta)\Theta\tilde{W} - \eta\Theta\tilde{W} + S(Z)z - \Theta W^*] \\ &\leq -(1-\eta)\lambda_{min}(\Theta)\|\Gamma\|\|\tilde{W}\|^2 \end{aligned} \tag{39}$$

for all $\|\tilde{W}\| \geq \frac{s^*\|z\|}{\eta\lambda_{min}(\Theta)} + \frac{\|W^*\|}{\eta}$, where $0 < \eta < 1$ is a constant. Therefore, system (37) is ISS with respect to inputs (z, W^*), with gain functions

$$\gamma_2^z(r) = \frac{s^*}{\eta\lambda_{min}(\Theta)}r, \quad \gamma_2^{W^*}(r) = \frac{1}{\eta}r. \qquad \diamondsuit$$

C. Stability of Closed-loop System Using Small-gain Theorem

From Section III-A and Section III-B, we know that that the closed-loop system consists of two interconnected subsystems: the state error subsystem and the weight estimation subsystem when we apply the controller (27) and the NN weight estimator (36) to the plant (1). Moreover, the state error subsystem is ISS with respect to the neural weight estimation errors and the weight estimation subsystem is ISS with respect to the system state errors. In the following, we will achieve the stability of the entire closed-loop system using the small-gain theorem.

Theorem 1: Consider the closed-loop system consisting of plant (1), reference model (2), controller (27) and NN weight estimator (36). Then, for bounded initial conditions, all signals in the closed-loop system remain bounded, and the output tracking error $y - y_d$ converges to a neighborhood of zero.

Proof: From Lemmas 2 and 3, we have obtained the ISS property of the state error subsystem (29) and the weight estimation subsystem (37). Therefore, there exist class $\mathcal{KL}$ function β_z and $\beta_{\tilde{W}}$, and class $\mathcal{K}$ functions $\gamma_1^{\tilde{W}}$, γ_1^{ε}, γ_2^z and $\gamma_2^{W^*}$, such that

$$\begin{aligned} \|z(\cdot)\|_\infty &\leq \max\{\beta_z(\|z(0)\|), \gamma_1^{\tilde{W}}(\|\tilde{W}\|_\infty), \gamma_1^{\varepsilon}(\|\varepsilon\|_\infty)\} \\ \|\tilde{W}(\cdot)\|_\infty &\leq \max\{\beta_{\tilde{W}}(\|\tilde{W}(0)\|), \gamma_2^z(\|z\|_\infty), \\ &\qquad \gamma_2^{W^*}(\|W^*\|_\infty)\} \end{aligned} \tag{40}$$

According to the small-gain theorem [15], by checking the following condition

$$\gamma_2^z(\gamma_1^{\tilde{W}}(r)) < r$$

we have

$$\frac{s^*}{\eta\lambda_{min}(\Theta)} \frac{s^*\sqrt{\bar{g}_*}}{\sqrt{2c_{*1}c_{*2}\underline{g}_*}} r < r$$

If we choose $c_{*1} = c_{*2}$, we obtain

$$c_{*1} > \frac{\sqrt{\bar{g}_*}s^{*2}}{\sqrt{2\underline{g}_*}\eta\lambda_{min}(\Theta)}$$

Then, given bounded initial conditions $z(0)$ and $\tilde{W}(0)$, we have

$$\begin{aligned} \|z(\cdot)\|_\infty &\leq \max\{\beta_z(\|z(0)\|), \gamma_1^{\tilde{W}}\beta_{\tilde{W}}(\|\tilde{W}(0)\|), \\ &\qquad \gamma_1^{\varepsilon}(\|\varepsilon\|_\infty), \gamma_1^{\tilde{W}}\gamma_2^{W^*}(\|W^*\|_\infty)\} \\ \|\tilde{W}(\cdot)\|_\infty &\leq \max\{\beta_{\tilde{W}}(\|\tilde{W}(0)\|), \gamma_2^z\beta_z(\|z(0)\|), \\ &\qquad \gamma_2^z\gamma_1^{\varepsilon}(\|\varepsilon\|_\infty), \gamma_2^{W^*}(\|W^*\|_\infty)\} \end{aligned} \tag{41}$$

which means that the closed-loop system is (locally) input-to-state stable [23] with respect to ε and W^*. Since $\|\varepsilon\| < \varepsilon^*$, and both ε^* and W^* are assumed to be constants, the boundedness of $(z, \tilde{W})$, and consequently, the boundedness of $x, \hat{W}$ and the control signal u can be established. Thus, all signals in the closed-loop remain bounded.

In addition,

$$\begin{aligned} \lim_{t\to\infty}\sup\|z(t)\| &\leq \max\{\gamma_1^{\varepsilon}(\|\varepsilon\|), \gamma_1^{\tilde{W}}\gamma_2^{W^*}(\|W^*\|)\} \\ &\leq \max\Big\{\frac{\sqrt{\bar{g}_*}}{\sqrt{2c_{*1}c_{*3}\underline{g}_*}}\varepsilon^*, \\ &\qquad \frac{s^*\sqrt{\bar{g}_*}}{\eta\sqrt{2c_{*1}c_{*2}\underline{g}_*}}\|W^*\|\Big\} \end{aligned} \tag{42}$$

which means that $\|z(t)\|$, including $z_1 = y - y_d$, will converge to a small neighborhood of zero by choosing c_i large enough. ■

IV. SIMULATION STUDIES

The following strict-feedback system is used for simulation studied in [11]

$$\begin{aligned} \dot{x}_1 &= 0.5x_1 + (1 + 0.1x_1^2)x_2 \\ \dot{x}_2 &= x_1x_2 + [2 + \cos(x_1)]u \\ y &= x_1 \end{aligned} \tag{43}$$

The control objective is to guarantee: i) all the signals in the closed-loop system remain bounded, and ii) the output y follows a desired trajectory y_d generated from the following van der Pol oscillator system:

$$\begin{aligned} \dot{x}_{d1} &= x_{d2} \\ \dot{x}_{d2} &= -x_{d1} + 0.2(1 - x_{d1}^2)x_{d2} \\ y_d &= x_{d1} \end{aligned} \tag{44}$$

As system (43) is of second order, the adaptive NN controller is chosen according to (27) as follows:

$$u = -z_1 - c_2z_2 - \hat{W}_2^T S_2(Z_2) \tag{45}$$

where $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$, and $Z_2 = [x_1, x_2, \partial\alpha_1/\partial x_1, \phi_1]^T$ with

$$\begin{aligned} \alpha_1 &= -c_1z_1 - \hat{W}_1^T S_1(Z_1) \\ \phi_1 &= \frac{\partial\alpha_1}{\partial x_{d1}}\dot{x}_{d1} + \frac{\partial\alpha_1}{\partial x_{d2}}\dot{x}_{d2} + \frac{\partial\alpha_1}{\partial\hat{W}_1}\dot{\hat{W}}_1 \end{aligned} \tag{46}$$

where $Z_1 = [x_1, \dot{x}_{d1}]^T$, and NN weights are updated by (36).

Neural networks $\hat{W}_1^T S_1(Z_1)$ contains 25 nodes (i.e., $l_1 = 25$), with centers $\mu_l(l = 1, ..., l_1)$ evenly spaced in $[-4, 4] \times [-4, 4]$, and widths $\eta_l = 1.0(l = 1, ..., l_1)$. Neural networks

$\hat{W}_2^T S_2(Z_2)$ contains 256 nodes (i.e.,$l_2 = 256$), with centers $\mu_l (l = 1, ..., l_2)$ evenly spaced in $[-4, 4] \times [-4, 4] \times [-4, 4] \times [-4, 4]$, and widths $\eta_l = 1.0 (l = 1, ..., l_2)$. The design parameters of the above controller are $c_1 = c_2 = 5.0$, $\Gamma_1 = \Gamma_2 = diag 2.0$, $\sigma_1 = \sigma_2 = 0.1$. The initial weights $\hat{W}_1(0) = 0.0$, $\hat{W}_2(0) = 0.0$. The initial conditions $[x_1(0), x_2(0)]^T = [1.2, 1.0]^T$ and $[x_{d1}(0), x_{d2}(0)]^T = [1.5, 0.8]^T$.

Figs. 1 and 2 show the simulation results of applying controller (45) to system (43) for tracking desired signal y_d. From Fig. 1, we can see that the good tracking performance is achieved by the proposed control. The boundedness of NN weights $\hat{W}_1$ and $\hat{W}_2$ and control signal u are shown in Fig. 2.

V. Conclusion

An adaptive neural network control using "ISS-modular" approach has been presented for a general class of strict-feedback systems in this paper. The ISS-modularity of the interconnected control module and estimation module has been achieved and the stability of the entire closed-loop system was guaranteed by the small-gain theorem. The ISS-modular approach provides an effective way for controlling the general class of strict-feedback systems.

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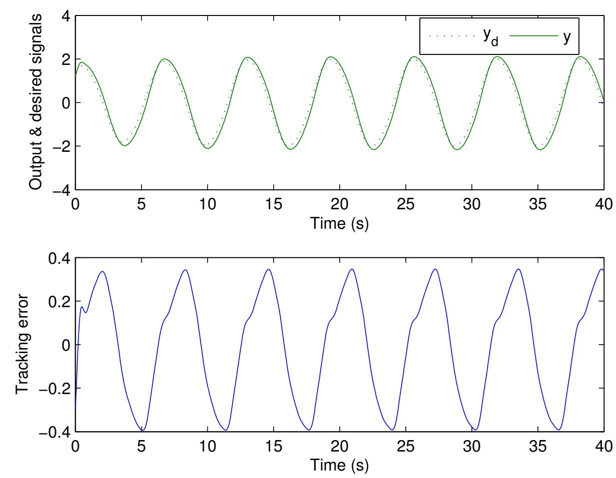

Fig. 1. Output tracking performance (Top) and tracking error (Bottom)

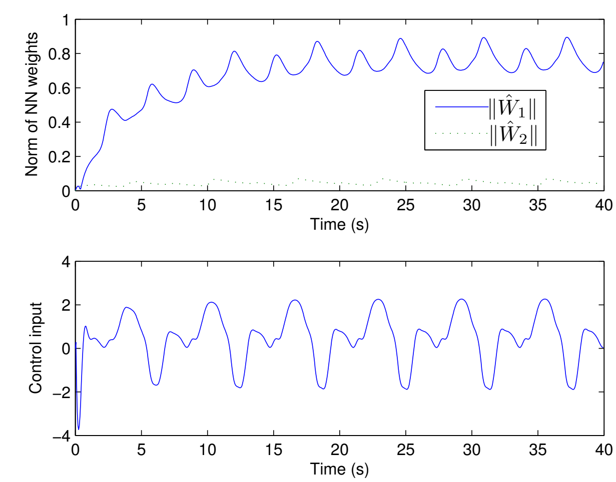

Fig. 2. Norm of NN weights (Top) and control input (Bottom)