UDE-based Robust Boundary Control of Heat Equation with Unknown Input Disturbance

Jiguo Dai * **Beibei Ren** *

* *Department of Mechanical Engineering, Texas Tech University, Lubbock, 79409-1021, USA. (e-mail: jiguo.dai@ttu.edu, beibei.ren@ttu.edu).*

Abstract: In this paper, a novel robust control strategy named uncertainty and disturbance estimator (UDE) is applied to the stabilization of an unstable heat equation with Dirichlet type boundary actuator and unknown time-varying input disturbance. The system is stabilized by the backstepping approach and the unknown input disturbance is compensated by the UDE-based method which constructs an estimation of the disturbance through filtering the system input $U(t)$ and boundary state $u(1,t)$. Compared to other existing disturbance compensation methods, the UDE-based method only requires the spectrum information of the disturbance signal. Furthermore, the output feedback version of the proposed control is also derived for the practical purpose. Stability analysis of the closed-loop system for both state feedback and output feedback cases are carried out and simulation examples are also provided to verify the proposed method.

Keywords: Heat equation, Boundary control, Input disturbance, Uncertainty and disturbance estimator.

1. INTRODUCTION

The boundary control for systems which are described by partial differential equations (PDEs) has drawn a lot of attention in control communities during the past decades. Numerous methods have been developed for this type of control problem in Krstic and Smyshlyaev (2008); Luo et al. (2012); Guo and Jin (2013a); Liu (2003), and the references therein. These systems usually describe the physical processes such as heat conduction, wave propagation or flexible beam, etc. Among these, the systems of heat conduction which are modeled by a parabolic type of PDEs (heat equation) are regarded as the simplest cases and the results for them can be extended to more other complicated PDE systems .

In Liu and Wang (2015), without considering the uncertainties or disturbances, control methods of the stabilization for the PDE systems can be divided into two categories, the collocated and non-collocated methods. The collocated methods follow the passive principle to make the closed-loop system dissipative in He et al. (2014); He and Ge (2015a,b), which assures the system stability in the sense of Lyapunov. The non-collocated methods are systematically applied to stabilize different PDE systems which can be unstable in Bošković et al. (2001); Liu (2003); Liu and Wang (2015) or even anti-stable in Smyshlyaev and Krstic (2009); Krstic (2010), by adopting the backstepping transformation in Krstic and Smyshlyaev (2008). However, the uncertainties and disturbances can not be negligible which demands the new methods for the practical applications.

To deal with the uncertainties and disturbances in the system, several methods have been widely used. Sliding mode control (SMC) which inherits a good robust performance has been applied to stabilize heat, wave, Euler-Bernoulli and Schrodinger equations with boundary input disturbances in Cheng et al. (2011); Guo and Jin (2013b,a); Guo and Liu (2014). In Liu and Wang (2015), the sliding mode controller is developed through the backstepping transformation for one-dimensional unstable heat equation with boundary uncertainties. Similarly, a sliding mode boundary controller for a heat equation with both parameter variations and boundary disturbances is developed in Cheng et al. (2011). In addition, an adaptive controller based on the Lyapunov method is derived for one-dimensional Euler-Bernoulli equations with both spatial and boundary disturbances in Ge et al. (2011). In Guo and Guo (2013), adaptive controllers for wave equations with unknown parameter and disturbance are proposed through the backstepping method. Furthermore, adaptive boundary control for heat equations with uncertain control coefficient and boundary disturbance is investigated in Li and Liu (2012). But it requires that the boundary disturbance vanishes while the time goes to infinity and the sign of the control coefficient should be known. Another robust method which is called active disturbance rejection control (ADRC) is also applied to the PDE systems, e.g., wave, Euler-Bernoulli and Schrodinger type PDE systems with boundary disturbance in Guo and Jin (2013a,b); Guo and Liu (2014). ADRC is also investigated in Liu and Wang (2015); Liu et al. (2015) for the unstable heat equations. The basic idea of ADRC is that an extended state observer (ESO) is derived through the system output and then ESO provides an estimation of the disturbance for compensation. However, the aforementioned methods have their own requirements, e.g., sliding mode control requires the knowledge of bounds of the disturbance; ADRC compensates the disturbance through an ESO, but in order to derive the ESO, a specific system output should be constructed as shown in Liu and Wang (2015); adaptive control sometimes has specific requirements for the disturbance.

In recent years, another method to handle the uncertainties and disturbances which is called uncertainty and disturbance estimator (UDE)-based control, has received more and more attention as shown in Zhong and Rees (2004); Zhong et al. (2011); Deshpande and Phadke (2012); Ren et al. (2015b); Kuperman and Zhong (2010). UDE was first proposed in Zhong and Rees (2004), and it compensates the disturbance through an estimation by filtering the system input and state information. Due to the simple structure and convenient implementation in real time, UDE has been applied to many linear, nonlinear systems in Deshpande and Phadke (2012); Ren et al. (2015b); Kuperman and Zhong (2010); Ren et al. (2015a); Wang et al. (2016); Ren et al. (2016), but it has not yet been applied to a PDE system. This paper considers the UDE-based boundary control for an unstable heat equation with unknown input disturbance. Specifically, the stabilization of the system is achieved through the backstepping approach and the boundary input disturbance is compensated by the UDE. The superiority of the UDE-based method is revealed from the disturbance estimation. Firstly, only the spectrum information of the disturbance is required. Furthermore, since the disturbance estimation is derived from the PDE boundary condition directly (i.e., the system input and boundary state), the construction of an additional system output is not required compared to ADRC.

The rest of this paper is organized as follows. The problem is formulated in Section 2. The UDE-based boundary controller is derived in Section 3 with the measurement of all the system states. An output feedback version of the UDE-based boundary control is provided in Section 4. Simulation examples are presented in Section 5 to show the effectiveness of the proposed control. Finally, Section 6 gives some conclusion remarks.

Notations: For notation convenience, in what follows we denote the derivatives $u_x(x,t) = \frac{\partial u(x,t)}{\partial x}$, $u_{xx}(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$, $u_t(x,t) = \frac{\partial u(x,t)}{\partial t}$, the L_2-norm of a continuous function $f(t)$ in $[0,1]$ is denoted as $||f(t)||_2 = \sqrt{\int_0^1 [f(t)]^2 dt}$.

2. PROBLEM FORMULATION

The heat conduction problem in a thin rod where the heat is being both diffused and destabilized can be modeled by the following equation:

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t),\ 0 \leq x \leq 1, t > 0 \quad (1)$$
$$u(x,0) = u_0,\ 0 \leq x \leq 1 \quad (2)$$

where $u(x,t) \in [0,1] \times (0,\infty)$ is the temperature profile at the spatial position x and the time t; u_0 is the initial condition; $\lambda > \frac{\pi^2}{4}$ is the destabilizing coefficient which indicates that the system is open-loop unstable. The boundary conditions are as follows

$$\textit{B.C.}:\ u_x(0,t) = 0,\ u(1,t) = U(t) + d(t) \quad (3)$$

where $U(t)$ is the boundary control input which is to be designed to stabilize the system, $d(t)$ is an unknown time-varying external disturbance. The boundary condition asserts that the control input only goes to the right end ($x = 1$) and the left end ($x = 0$) is assumed to be insulated. The external disturbance $d(t)$ is unknown and bounded, i.e., $|d(t)| \leq \bar{d}$, where $\bar{d}$ is the upper bound but it is not necessary to be known. Furthermore, it should be mentioned that the initial condition should be compatible with the boundary condition, i.e., $u_0(1) = U(0) + d(0)$. In addition, $u_x(1,t)$ is reasonably assumed to be bounded.

3. THE UDE-BASED BOUNDARY CONTROL: STATE FEEDBACK

3.1 Controller Design

Following the procedure of backstepping design in Krstic and Smyshlyaev (2008), the below invertible Volterra integral operator (backstepping transformation) is adopted to transform the original system into a target system,

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy \quad (4)$$

where $k(x,y)$ is known as the kernel function. The derivatives of $w(x,t)$ are calculated as

$$\begin{aligned} w_t(x,t) &= u_{xx}(x,t) + \lambda u(x,t) - k(x,x)u_x(x,t) \\ &+ k_y(x,x)u(x,t) - k_y(x,0)u(0,t) \\ &- \lambda \int_0^x k(x,y)u(y,t)dy - \int_0^x k_{yy}(x,y)u(y,t)dy \end{aligned} \quad (5)$$

and

$$w_x(x,t) = u_x(x,t) - k(x,x)u(x,t) - \int_0^x k_x(x,y)u(y,t)dy \quad (6)$$

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t) - u(x,t)\frac{d}{dx}k(x,x) - k(x,x)u_x(x,t) \\ &- k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)dy \end{aligned} \quad (7)$$

Combining (5), (6) and (7), there is,

$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) + cw(x,t) = -k_y(x,0)u(0,t) + \\ [\lambda + 2\frac{d}{dx}k(x,x) + c]u(x,t) + \int_0^x [k_{xx}(x,y) - k_{yy}(x,y) \\ -(\lambda + c)k(x,y)]u(y,t)dy \end{aligned} \quad (8)$$

where $c > 0$ is a design parameter to determine the convergence rate. In order to obtain the target system (9)-(11),

$$w_t(x,t) = w_{xx}(x,t) - cw(x,t) \quad (9)$$
$$w_x(0,t) = 0 \quad (10)$$
$$w(1,t) = U(t) + d(t) - \int_0^1 k(1,y)u(y,t)dy \quad (11)$$

let the right hand side of (8) be 0 and $\mu = \lambda + c > 0$. It yields that the kernel $k(x,y)$ should satisfy the following conditions,

$$k_{xx}(x,y) - k_{yy}(x,y) = \mu k(x,y) \quad (12)$$
$$k_y(x,0) = 0 \quad (13)$$
$$k(x,x) = -\frac{\mu}{2}x \quad (14)$$

Based on the successive approximation method, the kernel $k(x,y)$ can be obtained as in Krstic and Smyshlyaev (2008)

$$k(x,y) = -\mu x \frac{I_1(\sqrt{\mu(x^2 - y^2)})}{\sqrt{\mu(x^2 - y^2)}} \quad (15)$$

where I_1 is a first-order modified Bessel function in Krstic and Smyshlyaev (2008). The boundary controller is chosen as

$$U(t) = \int_0^1 k(1,y)u(y,t)dy - \hat{d}(t) \quad (16)$$

where the first term is to stabilize the system and the second term is to attenuate the disturbance based on its estimation. In order to estimate the disturbance, we follow the idea of UDE-based method in Zhong and Rees (2004). According to the equations (3) and (11),

$$\begin{aligned} d(t) &= w(1,t) + \int_0^1 k(1,y)u(y,t)dy - U(t) \\ &= u(1,t) - U(t) \end{aligned} \quad (17)$$

the disturbance can be estimated by introducing a low-pass filter

$$\begin{aligned}\hat{d}(t) &= \mathcal{L}^{-1}\{G_f(s)\} * d(t) \\ &= \mathcal{L}^{-1}\{G_f(s)\} * [u(1,t) - U(t)]\end{aligned} \tag{18}$$

where $\mathcal{L}^{-1}$ is the inverse Laplace operator, $*$ is the convolution operator and $G_f(s)$ is a low-pass filter in the frequency domain. Generally speaking, the $G_f(s)$ should be chosen to have an unity gain and zero phase shift within the spectrum of $d(t)$ as shown in Zhong and Rees (2004). Substituting $\hat{d}(t)$ into (16), it has

$$U(t) = \int_0^1 k(1,y)u(y,t)dy - \mathcal{L}^{-1}\{G_f(s)\} * [u(1,t) - U(t)] \tag{19}$$

Solving for $U(t)$ results in the UDE-based boundary controller

$$\begin{aligned}U(t) &= \mathcal{L}^{-1}\left\{\frac{1}{1-G_f(s)}\right\} * \int_0^1 k(1,y)u(y,t)dy \\ &-\mathcal{L}^{-1}\left\{\frac{G_f(s)}{1-G_f(s)}\right\} * u(1,t)\end{aligned} \tag{20}$$

3.2 Stability Analysis

The following assumption is required for the filter $G_f(s)$, while applying the UDE-based method.

Assumption 1. For a bounded disturbance signal $d(t)$, the filter $G_f(s)$ is designed with the unity gain and zero phase shift in the spectrum of $d(t)$, i.e., $\mathcal{L}^{-1}\{G_f(s)\} * d(t) = d(t)$ where $s \in Spectrum(d(t))$.

Remark 1. The Assumption 1 illustrates that if the filter is well designed, $\tilde{d}(t) = d(t) - \hat{d}(t) = d(t) - \mathcal{L}^{-1}\{G_f(s)\} * d(t) \to 0$ as $t \to \infty$. Therefore, the filter design plays an important role in the UDE-based method. The filter can be chosen as $G_f(s) = \frac{1}{Ts+1}$ while disturbance $d(t)$ is an unknown constant. For a more general type of disturbance, the selection of the filter $G_f(s)$ will be discussed later.

Then the following theorem asserts the stability of the closed-loop system.

Theorem 1. Consider the closed-loop system which consists of the system (1), the initial condition (2), the boundary condition (3), and the UDE-based boundary control law (20). If the filter $G_f(s)$ satisfies the Assumption 1, $\lim_{t\to\infty} ||u(x,t)||_2 = 0$.

Proof: Substituting the UDE-based boundary control law (16) into the system boundary condition (11), the closed-loop system is obtained as

$$w_t(x,t) = w_{xx}(x,t) - cw(x,t) \tag{21}$$

$$w_x(0,t) = 0 \tag{22}$$

$$w(1,t) = \tilde{d}(t) \tag{23}$$

where $\tilde{d}(t) = d(t) - \hat{d}(t)$. Choose the Lyapunov function candidate,

$$V(t) = \frac{1}{2}\int_0^1 w^2(x,t)dx \tag{24}$$

By applying the integration by parts, the derivative of $V(t)$ is given by

$$\begin{aligned}\dot{V}(t) &= \int_0^1 w(x,t)w_t(x,t)dx \\ &= \int_0^1 w(x,t)[w_{xx}(x,t) - cw(x,t)]dx \\ &= w_x(1,t)w(1,t) - w_x(0,t)w(0,t) - \int_0^1 w_x^2(x,t)dx \\ &- c\int_0^1 w^2(x,t)dx \\ &= w_x(1,t)\left[U(t) + d(t) - \int_0^1 k(1,y)u(y,t)dy\right] \\ &- \int_0^1 w_x^2(x,t)dx - c\int_0^1 w^2(x,t)dx \\ &= w_x(1,t)\tilde{d}(t) - \int_0^1 w_x^2(x,t)dx - c\int_0^1 w^2(x,t)dx\end{aligned} \tag{25}$$

According to the Assumption 1, there is $\tilde{d}(t) \to 0$ as $t \to \infty$. And $w_x(1,t) = u_x(1,t)$ is assumed to be bounded. Therefore,

$$\dot{V}(t) = -\int_0^1 w_x^2(x,t)dx - c\int_0^1 w^2(x,t)dx \tag{26}$$

By applying the Poincare Inequality (Lemma 2.1 in Krstic and Smyshlyaev (2008)), there is

$$\begin{aligned}\dot{V}(t) &\le -(c+\frac{1}{4})\int_0^1 w^2(x,t)dx \\ &= -(2c+\frac{1}{2})V(t)\end{aligned} \tag{27}$$

It indicates that $V(t) \le V(0)e^{-(2c+\frac{1}{2})t}$, which results in

$$||w(x,t)||_2 \le ||w(x,0)||_2 e^{-(c+\frac{1}{4})t} \tag{28}$$

Therefore, $\lim_{t\to\infty} ||w(x,t)||_2 = 0$ and the design parameter c is used for tuning the convergence rate. Then $\lim_{t\to\infty} \|u(x,t)\|_2 = 0$.□

3.3 Selection of the Filter $G_f(s)$

Assumption 2. The external boundary disturbance is assumed to have the form

$$d(t) = \sum_{i=0}^{k_t} d_i t^i + \sum_{j=0}^{k_s} \sigma_j \sin(\omega_j t + \phi_j) \tag{29}$$

where $k_t > 0$, $k_s > 0$ are known integers, d_i, σ_j and ϕ_j are unknown constants while the frequencies ω_j are known.

The Laplace transform of the disturbance component is

$$d(s) = \sum_{i=0}^{k_t} \frac{d_i^*}{s^{i+1}} + \sum_{j=0}^{k_s} \frac{\bar{\sigma}_j s + \tilde{\sigma}_j}{(s^2 + \omega_j^2)} \tag{30}$$

$$= \frac{q(s)}{s^{k_t+1}\Pi_{j=0}^{k_s}(s^2+\omega_j^2)} \tag{31}$$

where $d_i^*, \bar{\sigma}_j, \tilde{\sigma}_j$ are some constants, $q(s)$ is a polynomial of s. In order to guarantee the Assumption 1, let $G_f(s) = M(s)/N(s)$, while $N(s)$ is Hurwitz polynomial. Since the polynomial $1/\left(s^{k_t+1}\Pi_{j=0}^{k_s}(s^2+\omega_j^2)\right)$ is the internal model of the disturbance $d(t)$, it is sufficient to find the coefficients to satisfy the $N(s) - M(s) = s^{k_t+1}\Pi_{j=0}^{k_s}(s^2+\omega_j^2)$ and

$N(s)$ is Hurwitz polynomial with a suitable degree. Then $(1 - G_f(s))d(s) = \frac{N(s)-M(s)}{N(s)}d(s) = \frac{q(s)}{N(s)}$. Furthermore, it is easily verified that $\lim_{s\to 0} s(1 - G_f(s))d(s) = \frac{sq(s)}{N(s)} = 0$, then $\lim_{t\to\infty} \tilde{d}(t) = 0$ which is obtained from the final value theorem. Consequently, the Assumption 1 is satisfied. For details about the filter design to improve the performance of the UDE-based method, please refer to Ren et al. (2016). In Ren et al. (2016), the design of the filter uses the internal model principle to achieve the asymptotically rejection; just like the classical PI controllers could asymptotically reject the constant type disturbance as shown in Lamare and Bekiaris-Liberis (2015).

4. THE UDE-BASED BOUNDARY CONTROL: OUTPUT FEEDBACK

Since the state feedback requires the measurements of continuously distributed temperature along the bar, it restricts the applicability in the practical applications. Therefore, an output feedback version of the UDE-based boundary control is derived by using a state observer in this section.

Assumption 3. The sensor and actuator of the system (1) are anti-collocated, i.e., $u(0,t)$ and $u(1,t)$ are available for the measurements and $u(1,t)$ is the actuator.

4.1 Observer and Controller Design

Let $\hat{u}(x,t)$ denotes the estimation of the system state $u(x,t)$ and the following state observer is designed for the system (1), which follows from Smyshlyaev and Krstic (2005); Krstic and Smyshlyaev (2008),

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \lambda\hat{u}(x,t) + p_1(x)[u(0,t) - \hat{u}(0,t)] \tag{32}$$

$$\hat{u}_x(0,t) = p_{10}\left[u(0,t) - \hat{u}(0,t)\right] \tag{33}$$

$$\hat{u}(1,t) = U(t) + \hat{d}(t) \tag{34}$$

where $p_1(s)$ and p_{10} are the observer gains which need to be determined; $\hat{d}(t)$ is the input disturbance estimation obtained by the UDE-based method in (18) ; and $\hat{u}(x,0) = u_0$. Combining (1)-(3) and (32)-(34), and let $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$ be the observation error, the observation error system is

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + \lambda\tilde{u}(x,t) - p_1(x)\tilde{u}(0,t) \tag{35}$$

$$\tilde{u}_x(0,t) = -p_{10}\tilde{u}(0,t) \tag{36}$$

$$\tilde{u}(1,t) = \tilde{d}(t) \tag{37}$$

Let $\tilde{w}(x,t)$ be the transferred observation error, and consider the invertible backstepping transform $\tilde{u}(x,t) = \tilde{w}(x,t) - \int_0^x p(x,y)\tilde{w}(y,t)dy$ which is expected to transform the observation error system (35)-(37) into the following target system

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - \gamma\tilde{w}(x,t) \tag{38}$$

$$\tilde{w}_x(0,t) = 0 \tag{39}$$

$$\tilde{w}(1,t) = \tilde{d}(t) \tag{40}$$

where $\gamma > 0$ is the tuning parameter which determines the convergence rate. Let $\eta = \lambda + \gamma > 0$. As shown in Krstic and Smyshlyaev (2008), the observer gains are given as

$$p_1(x) = \frac{\eta(1-x)}{x(2-x)} I_2\left(\sqrt{\eta x(2-x)}\right) \tag{41}$$

$$p_{10} = -\frac{\eta}{2} \tag{42}$$

where I_2 is the second order modified Bessel function. Let $\hat{w}(x,t)$ denote the transferred system state, and the controller backstepping transform becomes

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_0^x k(x,y)\hat{u}(y,t)dy \tag{43}$$

where $k(x,y)$ is the kernel defined in (15). The original system will be transferred into the following target system

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \left\{p_1(x) - \int_0^x k(x,y)p_1(y)dy\right\}\tilde{w}(0,t) - c\hat{w}(x,t) \tag{44}$$

$$\hat{w}_x(0,t) = p_{10}\tilde{w}(0,t) \tag{45}$$

$$\hat{w}(1,t) = 0 \tag{46}$$

where c is the tuning parameter as defined in (9). Following the same idea of the UDE-based method, the UDE-based boundary controller has the same form as (16) or (20) while $u(y,t)$ will be replaced by the estimation $\hat{u}(y,t)$, i.e.,

$$U(t) = \int_0^1 k(1,y)\hat{u}(y,t)dy - \hat{d}(t) \tag{47}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{1-G_f(s)}\right\} * \int_0^1 k(1,y)\hat{u}(y,t)dy - \mathcal{L}^{-1}\left\{\frac{G_f(s)}{1-G_f(s)}\right\} * \hat{u}(1,t) \tag{48}$$

4.2 Stability Analysis

Theorem 2. Consider the closed-loop system which consists of the system (1), the observer (32)-(34), the initial condition (2), the boundary condition (3), and the UDE-based boundary control law (48). If the filter $G_f(s)$ satisfies the Assumption 1, then $\lim_{t\to\infty} \|\tilde{u}(x,t)\|_2 = 0$ and $\lim_{t\to\infty} \|\hat{u}(x,t)\|_2 = 0$.

Proof: The closed-loop system consists of the observation error system (38)-(40) and the target system (44)-(46). Choose the weighted Lyapunov function candidate,

$$V(t) = \frac{A}{2}\int_0^1 \tilde{w}^2(x,t)dx + \frac{1}{2}\int_0^1 \hat{w}^2(x,t)dx \tag{49}$$

where $A > 0$ is a weighting constant to be chosen later. By applying the integration by parts, the derivative of $V(t)$ is

$$\dot{V}(t) = A\tilde{d}(t)\tilde{w}_x(1,t) - A\int_0^1 \tilde{w}_x^2(x,t)dx - A\gamma\int_0^1 \tilde{w}^2(x,t)dx - p_{10}\hat{w}(0,t)\tilde{w}(0,t) - c\int_0^1 \hat{w}^2(x,t)dx - \int_0^1 \hat{w}_x^2(x,t)dx \int_0^1 \hat{w}(x,t)\left[p_1(x) - \int_0^x k(x,y)p_1(y)dy\right]\tilde{w}(0,t)dx \tag{50}$$

Applying the Poincare and Young's inequalities, there is

$$-p_{10}\hat{w}(0,t)\tilde{w}(0,t) \le \frac{1}{4}\hat{w}^2(0,t) + p_{10}^2\tilde{w}(0,t) \le \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)dx + p_{10}^2\int_0^1 \tilde{w}_x^2(x,t)dx \tag{51}$$

and

$$\tilde{w}(0,t)\int_0^1 \hat{w}(x,t)\left\{p_1(x) - \int_0^x k(x,y)p_1(y)dy\right\}dx \le \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)dx + B^2\int_0^1 \tilde{w}_x^2(x,t)dx \tag{52}$$

where $B = \max_{x\in[0,1]} \left\{p_1(x) - \int_0^x k(x,y)p_1(y)dy\right\}$. Furthermore, according to the Assumption 1, there is $\tilde{d}(t) \to 0$ as $t \to \infty$. And $\tilde{w}_x(1,t)$ is bounded. Then under these estimations, it can be obtained that

$$\dot{V}(t) \leq -(A - B^2 - p_{10}^2) \int_0^1 \tilde{w}_x^2(x,t)dx - \frac{1}{2}\int_0^1 \hat{w}_x^2(x,t)dx$$

$$- A\gamma \int_0^1 \tilde{w}^2(x,t)dx - c\int_0^1 \hat{w}^2(x,t)dx \quad (53)$$

$$\leq -\frac{1}{4}(A - B^2 - p_{10}^2 + 4A\gamma)\int_0^1 \tilde{w}^2(x,t)dx$$

$$- \frac{1}{8}(1+8c)\int_0^1 \hat{w}^2(x,t)dx \quad (54)$$

Taking $A = (B^2 + p_{10}^2)$, there is

$$\dot{V}(t) \leq -\min\left\{2\gamma, \frac{1}{4} + 2c\right\} V(t) \quad (55)$$

It indicates that $V(t) \leq V(0)e^{-\min\{2\gamma,\frac{1}{4}+2c\}t}$, which results in $\lim_{t\to\infty} ||\tilde{w}(x,t)||_2 = \lim_{t\to\infty} ||\hat{w}(x,t)||_2 = 0$ and the design parameters γ and c can be tuned to determine the convergence rate. Thus $\lim_{t\to\infty} \|\tilde{u}(x,t)\|_2 = 0$ and $\lim_{t\to\infty} \|\hat{u}(x,t)\|_2 = 0$.□

5. SIMULATION RESULTS

In this section, two simulation cases are carried out to demonstrate the effectiveness of the proposed UDE-based boundary control. The system parameters in (1) and (9) are chosen to be the same in both cases, i.e., $\lambda = 0.3\pi^2$, $c = 15$. The initial condition is set as $u_0 = 2\sin(\pi x)$.

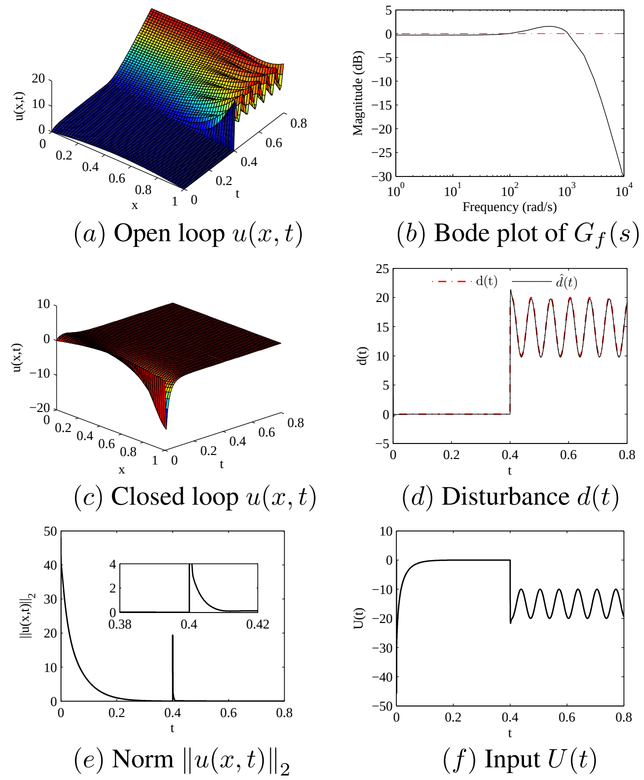
5.1 Case I: State feedback

In this case, the boundary input disturbance is chosen as, $d(t) = 15 + 5\sin(30\pi t + \pi/3)$, which is applied at $t = 0.4$s, as shown in Fig. 1(d). The filter $G_f(s)$ is designed as $G_f(s) = M(s)/N(s)$, where $M(s) = a_1 s^2 + (a_2 - \omega^2)s + a_3$, $N(s) = s^3 + a_1 s^2 + a_2 s + a_3$ with $a_1 = 2199$, $a_2 = 592180$, $a_3 = 1184$, and the frequency $\omega = 30\pi$. It can be seen that $N(s) - M(s) = s(s^2 + \omega^2)$, which is the internal model of the disturbance as mentioned in Section 3.3.

The simulation results are shown in Fig. 1. In Fig. 1(a), the open-loop system is unstable due to the destabilizing coefficient $\lambda = 0.3\pi^2 > \frac{\pi^2}{4}$ and has oscillations due to the input disturbance. Fig. 1(b) shown that the Bode plot of $G_f(s)$, which has 0dB magnitude within the spectrum of the disturbance signal. Figs. 1(c), (e) show that the closed-loop systems are stabilized because of the boundary control. Fig. 1(d) shows that the input disturbance is estimated by the filter approximately. Though a spike occurs, the controller still successfully mitigated the influence of the boundary input disturbance. The control input $U(t)$ is also shown in Fig. 1(f).

5.2 Case II: Output feedback

This case considers the same system as in the Case I, and a step type input boundary disturbance $d = 15$ is applied during 0.4s to 0.6s with the presence of the normal distribution random noise as shown in Fig. 2(d). The noise exists all the time after



(a) Open loop $u(x,t)$ (b) Bode plot of $G_f(s)$ (c) Closed loop $u(x,t)$ (d) Disturbance $d(t)$ (e) Norm $\|u(x,t)\|_2$ (f) Input $U(t)$

Fig. 1. Case I: State feedback

$t = 0.4$s. For the practical considerations, the output feedback approach is utilized and the first order low-pass filter is selected as $G_f(s) = \frac{1}{Ts+1}$ with $T = 0.002$. In addition, $\gamma = 2$ in (38).

The results are shown in Fig. 2. The unstable system can be stabilized by the UDE-based boundary control as shown in Fig. 2(a). The observer gain $p_1(x)$ is calculated through the system parameters and shown in Fig. 2(b). Fig. 2(c) presents the system output $u(0,t)$ and its estimate $\hat{u}(0,t)$. During 0.2s to 0.4s, though the step type disturbance with noise is trying to perturb the system, their influence is mitigated by the UDE-based boundary control effectively, as shown in Figs. 2(d) and (e). The corresponding control signal is shown in Fig. 2(f).

6. CONCLUSIONS

The UDE-based boundary control was developed for the unstable heat equation with unknown input disturbance in this paper. Both state feedback and output feedback cases were studied. The backstepping approach was applied to stabilize the unstable system and the boundary input disturbance was mitigated by the UDE. The proposed UDE-based boundary control only uses the system input and boundary state information to construct an estimation of the disturbance. Moreover, UDE inherits a simple structure and only requires the spectrum information of the disturbance. The stability analysis of the closed-loop systems and numerical examples were provided for validation.

REFERENCES

Bošković, D.M., Krstić, M., and Liu, W. (2001). Boundary control of an unstable heat equation via measurement of domain-averaged temperature. *IEEE Tran. on Automatic Control*, 46(12), 2022–2028.

Cheng, M.B., Radisavljevic, V., and Su, W.C. (2011). Sliding mode boundary control of a parabolic PDE system with pa-

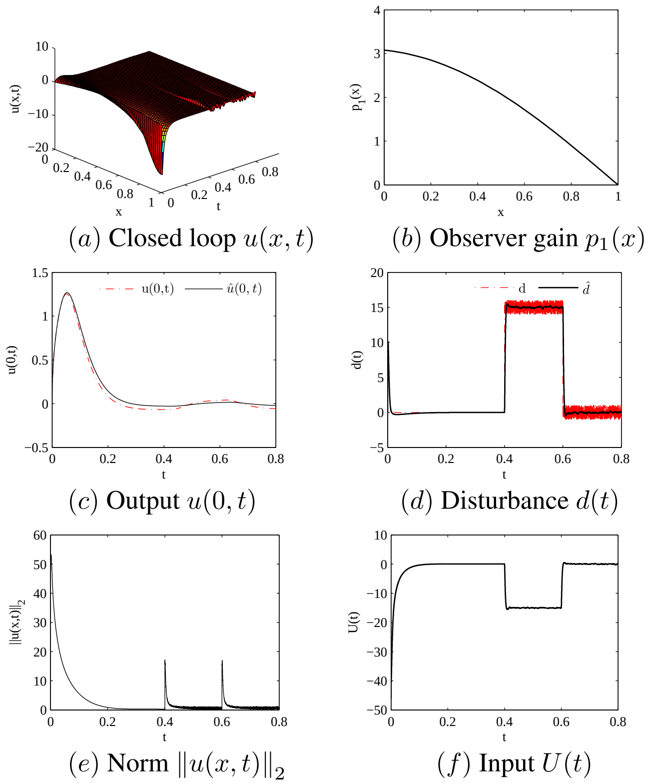


Fig. 2. Case II: Output feedback

rameter variations and boundary uncertainties. *Automatica*, 47(2), 381–387.

Deshpande, V. and Phadke, S. (2012). Control of uncertain nonlinear systems using an uncertainty and disturbance estimator. *Journal of Dyn. Sys. Meas. Con. Trans.*, 134(2), 024501.

Ge, S.S., Zhang, S., and He, W. (2011). Vibration control of an Euler–Bernoulli beam under unknown spatiotemporally varying disturbance. *International Journal of Control*, 84(5), 947–960.

Guo, B.Z. and Jin, F.F. (2013a). The active disturbance rejection and sliding mode control approach to the stabilization of the Euler–Bernoulli beam equation with boundary input disturbance. *Automatica*, 49(9), 2911–2918.

Guo, B.Z. and Jin, F.F. (2013b). Sliding mode and active disturbance rejection control to stabilization of one-dimensional anti-stable wave equations subject to disturbance in boundary input. *IEEE Tran. on Automatic Control*, 58(5), 1269–1274.

Guo, B.Z. and Liu, J.J. (2014). Sliding mode control and active disturbance rejection control to the stabilization of one-dimensional Schrödinger equation subject to boundary control matched disturbance. *International Journal of Robust and Nonlinear Control*, 24(16), 2194–2212.

Guo, W. and Guo, B.Z. (2013). Parameter estimation and non-collocated adaptive stabilization for a wave equation subject to general boundary harmonic disturbance. *IEEE Tran. on Automatic Control*, 58(7), 1631–1643.

He, W. and Ge, S.S. (2015a). Vibration control of a flexible beam with output constraint. *IEEE Tran. on Industrial Electronics*, 62(8), 5023–5030.

He, W. and Ge, S.S. (2015b). Vibration control of a flexible string with both boundary input and output constraints. *IEEE Tran. on Control Systems Technology*, 23(4), 1245–1254.

He, W., Zhang, S., and Ge, S.S. (2014). Adaptive boundary control of a nonlinear flexible string system. *IEEE Tran. on Control Systems Technology*, 22(3), 1088–1093.

Krstic, M. (2010). Adaptive control of an anti-stable wave PDE. *Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis*, 17, 853–882.

Krstic, M. and Smyshlyaev, A. (2008). *Boundary control of PDEs: A course on backstepping designs*. SIAM.

Kuperman, A. and Zhong, Q.C. (2010). Robust control of uncertain nonlinear systems with state delays based on an uncertainty and disturbance estimator. *International Journal of Robust and Nonlinear Control*, 21(1), 79–92.

Lamare, P.O. and Bekiaris-Liberis, N. (2015). Control of 2× 2 linear hyperbolic systems: Backstepping-based trajectory generation and pi-based tracking. *Systems & Control Letters*, 86, 24–33.

Li, J. and Liu, Y.G. (2012). Adaptive boundary control for heat equations with uncertain control coefficient and boundary disturbance. *Acta Automatica Sinica*, 38(3), 469–473.

Liu, J., Zheng, G., and Ali, M.M. (2015). Stability analysis of the anti-stable heat equation with uncertain disturbance on the boundary. *Journal of Mathematical Analysis and Applications*, 428(2), 1193–1201.

Liu, J.J. and Wang, J.M. (2015). Active disturbance rejection control and sliding mode control of one-dimensional unstable heat equation with boundary uncertainties. *IMA Journal of Mathematical Control and Information*, 32(1), 97–117.

Liu, W. (2003). Boundary feedback stabilization of an unstable heat equation. *SIAM Journal on Control and Optimization*, 42(3), 1033–1043.

Luo, Z.H., Guo, B.Z., and Morgül, Ö. (2012). *Stability and stabilization of infinite dimensional systems with applications*. Springer Science & Business Media.

Ren, B., Wang, Y., and Zhong, Q.C. (2015a). UDE-based control of variable-speed wind turbine systems. *International Journal of Control*, 90(1), 137–152.

Ren, B., Zhong, Q.C., and Chen, J. (2015b). Robust control for a class of non-affine nonlinear systems based on the uncertainty and disturbance estimator. *IEEE Trans. Ind. Electron.*, 62(9), 5881–5888.

Ren, B., Zhong, Q.C., and Dai, J. (2016). Asymptotic reference tracking and disturbance rejection of UDE-based robust control. *IEEE Trans. Ind. Electron.* DOI 10.1109/TIE.2016.2633473.

Smyshlyaev, A. and Krstic, M. (2005). Backstepping observers for a class of parabolic PDEs. *Systems & Control Letters*, 54(7), 613–625.

Smyshlyaev, A. and Krstic, M. (2009). Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary. *Systems & Control Letters*, 58(8), 617–623.

Wang, Y., Ren, B., and Zhong, Q.C. (2016). Robust power flow control of grid-connected inverters. *IEEE Trans. Ind. Electron.*, 63(11), 6887–6897.

Zhong, Q.C., Kuperman, A., and Stobart, R. (2011). Design of UDE-based controllers from their two-degree-of-freedom nature. *International Journal of Robust and Nonlinear Control*, 21, 1994–2008.

Zhong, Q.C. and Rees, D. (2004). Control of uncertain LTI systems based on an uncertainty and disturbance estimator. *Journal of Dyn. Sys. Meas. Con.Trans. ASME*, 126(4), 905–910.