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Brief paper

Control of nonlinear systems with time-varying output constraints[☆]

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ABSTRACT

This paper presents control design for strict feedback nonlinear systems with time-varying output constraints. An asymmetric time-varying Barrier Lyapunov Function (BLF) is employed to ensure constraint satisfaction. By allowing the barriers to vary with the desired trajectory in time, the initial condition requirements are relaxed. Through a change of tracking error coordinates, we eliminate the explicit dependence of the BLF on time, thereby simplifying the analysis of constraint satisfaction. We show that asymptotic output tracking is achieved without violation of the output constraint, and also quantify the transient performance bound as a function of time that converges to zero. To handle parametric model uncertainty, we present an adaptive controller that ensures constraint satisfaction during the transient phase of online parameter adaptation. The performance of the proposed control is illustrated through a simulation example.

1. Introduction

Driven by practical needs and theoretical challenges, the rigorous handling of constraints in control design has become an important research topic. Constraint-handling methods based on set invariance (Hu & Lin, 2001; Liu & Michel, 1994), model predictive control (Mayne, Rawlings, Rao, & Scokaert, 2000) and reference governors (Bemporad, 1998; Gilbert & Ong, 2009) are well established. Other notable methods include extremum seeking control (DeHaan & Guay, 2005), nonovershooting control (Krstic & Bement, 2006), adaptive variable structure control (Su, Stepanenko, & Leung, 1995), and error transform (Do, 2010).

More recently, the use of Barrier Lyapunov Function (BLF) for control of nonlinear systems with output and state constraints has been proposed. BLFs have been used to design control for output-constrained systems in strict feedback form (Tee, Ge, & Tay, 2009b) and output feedback form (Ren, Ge, Tee, & Lee, 2010). Using an asymmetric barrier function allows relaxation of the initial condition requirements (Tee et al., 2009b). The BLF-based design framework accommodates adaptive control design for handling not only parametric uncertainty (Tee et al., 2009b), but also function uncertainty through the use of neural networks (Ren et al., 2010). BLF-based control design has also been used for state-constrained systems in Brunovsky form (Ngo, Mahony, & Jiang, 2005) and strict feedback form (Tee & Ge, 2009). In addition, BLF-based control has been applied to practical systems, such as electromagnetic oscillators (Sane & Bernstein, 2002), electrostatic parallel plate microactuators (Tee, Ge, & Tay, 2009a), and electrostatic torsional micromirrors (Zhu, Agudelo, Saydy, & Packirisamy, 2008).

Besides static constraints considered in the above-mentioned works involving BLFs, time-varying output constraints have also been tackled, by using a time-varying BLF (Tee, Ge, Li, & Ren, 2009), as well as multiple BLFs under a switching scheme (Yan & Wang, 2010). Different from Tee, Ge, Li, et al. (2009), which focused on a symmetric BLF to handle symmetric output constraints, this paper presents a generalization of the results based on an asymmetric BLF that can handle asymmetric output constraints. Furthermore, an adaptive version of the control is presented to deal with parametric model uncertainty. Through a change of tracking error coordinates, we eliminate the explicit dependence of the BLF on time, thus facilitating the adoption of an analysis framework similar to that of Tee et al. (2009b) for the static constraint problem. The advantages of the proposed control are summarized as follows:

(i) The control is able to handle an output constraint that is both time-varying and asymmetric.
(ii) When time-varying asymmetric BLF is used, the output can start from anywhere within the initial output constrained space.

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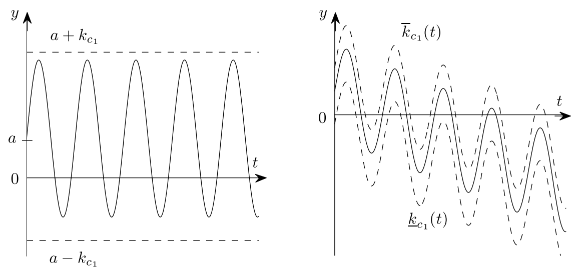

Fig. 1. Static (left) and asymmetric time-varying (right) constraints. Dashed lines represent the constraint boundaries.

(iii) For the known case, we quantify the decay of the bound for the error state z as a function of time that converges to zero.
(iv) For the known case with constant output error bounds, the origin of the closed loop error system is locally exponentially stable.
(v) In the absence of output constraint, the control allows shaping of the transient tracking error trajectory.

In what follows, Section 2 formulates the asymmetric time-varying output constraint problem. Then, Section 3 presents time-varying BLF-based control design for known and uncertain systems, and addresses practical issues pertaining to initial output conditions and robustness to disturbances. Lastly, Section 4 provides a simulation example that illustrates performance.

2. Problem formulation and preliminaries

Consider the strict feedback nonlinear system:

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad i = 1, 2, \dots, n-1$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u$$
$$y = x_1 \tag{1}$$

where $f_1, \dots, f_n, g_1, \dots, g_n$ are smooth functions, $x_1, \dots, x_n$ are the states, u and y are the input and output respectively, and $\bar{x}_i = [x_1, x_2, \dots, x_i]^T$. We deal with the class of time-varying asymmetric output constraints (Fig. 1), which is general, and include static (Tee et al., 2009b) and symmetric time-varying ones (Tee, Ge, Li, et al., 2009) as special cases. Specifically, the output $y(t)$ is required to satisfy

$$\underline{k}_{c_1}(t) < y(t) < \overline{k}_{c_1}(t), \quad \forall t \geq 0 \tag{2}$$

where $\overline{k}_{c_1}: \mathbb{R}_+ \to \mathbb{R}$ and $\underline{k}_{c_1}: \mathbb{R}_+ \to \mathbb{R}$ such that $\overline{k}_{c_1}(t) > \underline{k}_{c_1}(t)$ $\forall t \in \mathbb{R}_+$.

The control objective is to track a desired trajectory $y_d(t)$ while ensuring that all closed loop signals are bounded and that the *output constraint is not violated.*

Assumption 1. The functions $g_i(\bar{x}_i)$, $i = 1, 2, \dots, n$, are known, and there exists a positive constant g_0 such that $0 < g_0 \leq |g_i(\bar{x}_i)|$ for $y = x_1$ satisfying (2). Without loss of generality, we further assume that the $g_i(\bar{x}_i)$ are all positive.

Assumption 2. There exist constants $\overline{K}_{c_i}$ and $\underline{K}_{c_i}$ $i = 0, 1, \dots, n$, such that $\overline{k}_{c_1}(t) \leq \overline{K}_{c0}$, $\underline{k}_{c_1}(t) \geq \underline{K}_{c0}$ and $|\overline{k}_{c_1}^{(i)}(t)| \leq \overline{K}_{c_i}$, $|\underline{k}_{c_1}^{(i)}(t)| \leq \underline{K}_{c_i}$, $i = 1, \dots, n$, $\forall t \geq 0$.

Assumption 3. There exist functions $\overline{Y}_0: \mathbb{R}_+ \to \mathbb{R}_+$ and $\underline{Y}_0: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\overline{Y}_0(t) < \overline{k}_{c_1}(t)$ and $\underline{Y}_0(t) > \underline{k}_{c_1}(t)$ $\forall t \geq 0$, and positive constants Y_i, $i = 1, \dots, n$, such that the desired trajectory $y_d(t)$ and its time derivatives satisfy $\underline{Y}_0(t) \leq y_d(t) \leq \overline{Y}_0(t)$ and $|y_d^{(i)}(t)| \leq Y_i$, $i = 1, \dots, n$, $\forall t \geq 0$.

The following lemmata are useful for establishing constraint satisfaction and performance bounds.

Lemma 1 (Tee et al., 2009b). *Let $\mathcal{Z} := \{\xi \in \mathbb{R}: |\xi| < 1\} \subset \mathbb{R}$ and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z} \subset \mathbb{R}^{l+1}$ be open sets. Consider the system*

$$\dot{\eta} = h(t, \eta)$$

where $\eta := [w, \xi]^T \in \mathcal{N}$, and $h: \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{l+1}$ is piecewise continuous in t and locally Lipschitz in η, uniformly in t, on $\mathbb{R}_+ \times \mathcal{N}$. Suppose that there exist functions $U: \mathbb{R}^l \times \mathbb{R}_+ \to \mathbb{R}_+$ and $V_1: \mathcal{Z} \to \mathbb{R}_+$, continuously differentiable and positive definite in their respective domains, such that

$$V_1(\xi) \to \infty \quad \text{as } |\xi| \to 1$$
$$\gamma_1(\|w\|) \leq U(w, t) \leq \gamma_2(\|w\|)$$

where γ_1 and γ_2 are class K_∞ functions. Let $V(\eta) := V_1(\xi) + U(w, t)$, and $\xi(0) \in \mathcal{Z}$. If the inequality holds:

$$\dot{V} = \frac{\partial V}{\partial \eta} h \leq 0$$

in the set $\xi \in \mathcal{Z}$, then $\xi(t) \in \mathcal{Z}$ $\forall t \in [0, \infty)$.

Lemma 2 (Ren et al., 2010). *For all $|\xi| < 1$ and any positive integer p, the inequality $\log 1/(1-\xi^{2p}) < \xi^{2p}/(1-\xi^{2p})$ holds.*

3. Time-varying BLF-based control

To handle asymmetric time-varying output constraints, we employ asymmetric time-varying barrier functions, which can also handle symmetric output constraints (Tee, Ge, Li, et al., 2009), and static ones (Tee et al., 2009b). We present control designs for both known and uncertain versions of the plant, as well as practically-motivated discussions on initial output conditions and robustness to disturbances.

3.1. Control design for known system

The control design is based on backstepping with an asymmetric time-varying barrier function.

Step 1: Denote $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, where α_1 is a stabilizing function. Consider the time-varying asymmetric barrier function:

$$V_1 = \frac{q(z_1)}{2p} \log \frac{k_{b_1}^{2p}(t)}{k_{b_1}^{2p}(t) - z_1^{2p}} + \frac{1-q(z_1)}{2p} \log \frac{k_{a_1}^{2p}(t)}{k_{a_1}^{2p}(t) - z_1^{2p}} \tag{3}$$

where p is a positive integer satisfying $2p \geq n$ so as to ensure differentiability of the stabilizing functions α_i, $i = 1, \dots, n-1$. The time-varying barriers are given by

$$k_{a_1}(t) := y_d(t) - \underline{k}_{c_1}(t) \tag{4}$$

$$k_{b_1}(t) := \overline{k}_{c_1}(t) - y_d(t) \tag{5}$$

$$q(\bullet) := \begin{cases} 1, & \text{if } \bullet > 0 \\ 0, & \text{if } \bullet \leq 0. \end{cases} \tag{6}$$

Throughout this paper, for ease of notation, we abbreviate $q(z_1)$ by q, unless otherwise stated.

Due to Assumptions 2–3, there exist positive constants $\underline{k}_{b_1}$, $\overline{k}_{b_1}$, $\underline{k}_{a_1}$ and $\overline{k}_{a_1}$ such that

$$\underline{k}_{b_1} \leq k_{b_1}(t) \leq \overline{k}_{b_1}, \qquad \underline{k}_{a_1} \leq k_{a_1}(t) \leq \overline{k}_{a_1}, \quad \forall\, t \geq 0. \tag{7}$$

By a change of error coordinates

$$\xi_a = \frac{z_1}{k_{a_1}}, \qquad \xi_b = \frac{z_1}{k_{b_1}}, \qquad \xi = q\xi_b + (1-q)\xi_a \tag{8}$$

for $i = 1, \ldots, n$, we can rewrite (3) into a form that does not depend explicitly on time:

$$V_1 = \frac{1}{2p} \log \frac{1}{1 - \xi^{2p}}. \tag{9}$$

It is clear that V_1 is positive definite and continuously differentiable in the set $|\xi| < 1$. The time derivative of V_1 is given by

$$\dot{V}_1 = \frac{q\xi_b^{2p-1}}{k_{b_1}(1-\xi_b^{2p})} \left(f_1 + g_1(z_2 + \alpha_1) - \dot{y}_d - z_1 \frac{\dot{k}_{b_1}}{k_{b_1}} \right) + \frac{(1-q)\xi_a^{2p-1}}{k_{a_1}(1-\xi_a^{2p})} \left(f_1 + g_1(z_2 + \alpha_1) - \dot{y}_d - z_1 \frac{\dot{k}_{a_1}}{k_{a_1}} \right). \tag{10}$$

Design the stabilizing function α_1 as

$$\alpha_1 = \frac{1}{g_1} \left(-f_1 - (\kappa_1 + \bar{\kappa}_1(t)) z_1 + \dot{y}_d \right) \tag{11}$$

where the time-varying gain is given by

$$\bar{\kappa}_1(t) = \sqrt{\left(\frac{\dot{k}_{a_1}}{k_{a_1}}\right)^2 + \left(\frac{\dot{k}_{b_1}}{k_{b_1}}\right)^2 + \beta} \tag{12}$$

for any positive constants β and κ_1. Note that β ensures that the time derivatives of α_1 are bounded even when $\dot{k}_{a_1}$ and $\dot{k}_{b_1}$ are both zero. Substituting (8) and (11)–(12) into (10), and noting that

$$\bar{\kappa}_1 + q\frac{\dot{k}_{b_1}}{k_{b_1}} + (1-q)\frac{\dot{k}_{a_1}}{k_{a_1}} \geq 0 \tag{13}$$

we obtain

$$\dot{V}_1 \leq -\frac{\kappa_1 \xi^{2p}}{1 - \xi^{2p}} + \mu_1 g_1 z_1^{2p-1} z_2$$

where $\mu_1 := q/(k_{b_1}^{2p} - z_1^{2p}) + (1-q)/(k_{a_1}^{2p} - z_1^{2p})$.

Step i ($i = 2, \ldots, n$): Let $z_i = x_i - \alpha_{i-1}$, and consider the quadratic functions $V_i = z_i^2/2$, $i = 2, \ldots, n$. Design stabilizing functions and control law as

$$\begin{aligned} \alpha_2 &= \frac{1}{g_2}(-\kappa_2 z_2 - f_2 + \dot{\alpha}_1 - \mu_1 g_1 z_1^{2p-1}) \\ \alpha_i &= \frac{1}{g_i}(-\kappa_i z_i - f_i + \dot{\alpha}_{i-1} - g_{i-1} z_{i-1}), \quad i = 3, \ldots, n \\ u &= \alpha_n \end{aligned} \tag{14}$$

which yields the closed loop system

$$\begin{aligned} \dot{z}_1 &= -(\kappa_1 + \bar{\kappa}_1) z_1 + g_1 z_2 \\ \dot{z}_2 &= -\kappa_2 z_2 - \mu_1 g_1 z_1^{2p-1} + g_2 z_3 \\ \dot{z}_i &= -\kappa_i z_i - g_{i-1} z_{i-1} + g_i z_{i+1}, \quad i = 3, \ldots, n-1 \\ \dot{z}_n &= -\kappa_n z_n - g_{n-1} z_{n-1} \end{aligned} \tag{15}$$

where the right hand side is piecewise continuous in t and locally Lipschitz in z, uniformly in t. Then, we can show that the time derivative of $V = \sum_{i=1}^n V_i$ satisfies

$$\dot{V} \leq -\frac{\kappa_1 \xi^{2p}}{1 - \xi^{2p}} - \sum_{i=2}^n \kappa_i z_i^2. \tag{16}$$

Lemma 3. *The condition $|\xi(t)| < 1$ holds iff $-k_{a_1}(t) < z_1(t) < k_{b_1}(t)$.*

Proof. First, we show that $|\xi(t)| < 1 \Rightarrow -k_{a_1}(t) < z_1(t) < k_{b_1}(t)$. From (8), consider $z_1(t) \leq 0$ for some $t > 0$, which yields $-1 < \xi_a(t) \leq 0$. Since $\xi_a = z_1/k_{a_1}$ for $z_1 \leq 0$, and $k_{a_1} > 0$, we obtain $-k_{a_1}(t) < z_1(t) \leq 0$. Similarly, considering $z_1(t) > 0$ for some $t > 0$ yields $0 < \xi_b(t) \leq 1$ and, in turn, $0 < z_1(t) < k_{b_1}(t)$ Combining both cases, we conclude that $-k_{a_1}(t) < z_1(t) < k_{b_1}(t)$, $\forall\, t > 0$. To show that $-k_{a_1}(t) < z_1(t) < k_{b_1}(t) \Rightarrow |\xi(t)| < 1$ is straightforward by a reverse procedure. □

Theorem 1. *Consider the closed loop system (1), (11), (14), and Assumptions 1, 2, 3. If the initial output $y(0)$ satisfies $\underline{k}_{c_1}(0) < y(0) < \overline{k}_{c_1}(0)$, then the following properties hold.*

(i) *The error signals $z_i(t)$, $i = 1, 2, \ldots, n$, are bounded by*

$$-\underline{D}_{z_1}(t) \leq z_1(t) \leq \overline{D}_{z_1}(t), \qquad \|z_{2:n}(t)\| \leq D_{z_{2:n}}(t)$$

$\forall\, t > 0$, where the bounds $\underline{D}_{z_1}$, $\overline{D}_{z_1}$, and $D_{z_{2:n}}$ converge to zero as follows:

$$\begin{aligned} \overline{D}_{z_1}(t) &= k_{b_1}(t) \left(1 - e^{-2pV(0)e^{-\rho t}}\right)^{\frac{1}{2p}} \\ \underline{D}_{z_1}(t) &= k_{a_1}(t) \left(1 - e^{-2pV(0)e^{-\rho t}}\right)^{\frac{1}{2p}} \\ D_{z_{2:n}}(t) &= \sqrt{2V(0)e^{-\rho t}} \end{aligned} \tag{17}$$

with $\rho := \min\{2p\kappa_1, 2\kappa_2, \ldots, 2\kappa_n\}$ a positive constant.

(ii) *The asymmetric time-varying output constraint is never violated, i.e. $\underline{k}_{c_1}(t) < y(t) < \overline{k}_{c_1}(t)$, $\forall t > 0$.*

(iii) *All closed loop signals are bounded.*

Proof. (i) Based on the definitions of k_{a_1} and k_{b_1} in (4)–(5), we rewrite the initial condition requirement as $-k_{a_1}(0) < z_1(0) < k_{b_1}(0)$. This is equivalent to $|\xi(0)| < 1$, as follows from Lemma 3. Then, Lemma 1 ensures that $|\xi(t)| < 1\, \forall t > 0$.

From (16) and Lemma 2, we can show that $\dot{V}(t) \leq -\rho V(t)$, $\forall t > 0$, where $\rho = \min\{2p\kappa_1, 2\kappa_2, \ldots, 2\kappa_n\}$. Integrating both sides of the inequality yields $V(t) \leq V(0)e^{-\rho t}$. Thus, we have $(1/2p)\log(1/(1-\xi^{2p})) \leq V(0)e^{-\rho t}$, which leads to

$$\xi^{2p} \leq 1 - e^{-2pV(0)e^{-\rho t}}. \tag{18}$$

Based on the coordinate transformation (8), it is obvious that $-\underline{D}_{z_1}(t) \leq z_1(t) \leq \overline{D}_{z_1}(t)\, \forall t \geq 0$. Furthermore, from the fact that $\frac{1}{2}\sum_{j=2}^n z_j^2(t) \leq V(0)e^{-\rho t}$, we can show that $\|z_{2:n}(t)\| \leq \sqrt{2V(0)e^{-\rho t}}\, \forall\, t > 0$.

(ii) Since $|\xi(t)| < 1$, we know that $-k_{a_1}(t) < z_1(t) < k_{b_1}(t)$ from Lemma 3. Together with the fact that $y(t) = z_1(t) + y_d(t)$, we infer that

$$-k_{a_1}(t) + y_d(t) < y(t) < k_{b_1}(t) + y_d(t) \tag{19}$$

for all $t > 0$. From the definitions of k_{a_1} and k_{b_1} in (4) and (5) respectively, we conclude that $\underline{k}_{c_1}(t) < y(t) < \overline{k}_{c_1}(t)\, \forall t > 0$.

(iii) The error signals $z_i(t)$, $i = 1, \ldots, n$, and the state $x_1(t)$, are bounded, as shown in (i) and (ii). Using (7), we obtain constant bounds for z_1 as $-\underline{k}_{a_1} < z_1(t) < \overline{k}_{b_1}$, and we know that k_{a_1}, k_{b_1} are bounded away from 0. Furthermore, we estimate the bounds $|\dot{k}_{b_1}| \leq Y_1 + \overline{K}_{c_1}$ and $|\dot{k}_{a_1}| \leq Y_1 + \underline{K}_{c_1}$ from (4)–(5) and Assumptions 2–3.

Then, based on (11), we can show that the stabilizing function $\alpha_1(t)$ is bounded. This leads to boundedness of $x_2(t)$, from $x_2 = z_2 + \alpha_1$.

By signal chasing, we can progressively show that $\alpha_i(t)$, $i = 3, \ldots, n-1$, are bounded. Thus, the boundedness of state $x_{i+1}(t)$ can be shown. With $\bar{x}_n(t)$, $\bar{z}_n(t)$ bounded, and $|\xi(t)| < 1\, \forall\, t > 0$, we conclude that the control $u(t)$ is bounded. Hence, all closed loop signals are bounded. □

Corollary 1. *If k_{a_1} and k_{b_1} are constants, the origin of the closed loop system (15) is locally exponential stable.*

Proof. Since $1 - e^{-2pV(0)e^{-\rho t}} \leq 2pV(0)e^{-\rho t}$, it follows, from (17), that $z_1(t)$ is upper and lower bounded by exponentially decreasing functions $k_{b_1}\left(2pV(0)e^{-\rho t}\right)^{\frac{1}{2p}}$ and $-k_{a_1}\left(2pV(0)e^{-\rho t}\right)^{\frac{1}{2p}}$, respectively, $\forall t > 0$. Together with the fact that $\|z_{2:n}(t)\| \leq \sqrt{2V(0)e^{-\rho t}}$, we conclude that $z = 0$ is locally exponential stable. □

3.2. Handling parametric uncertainty

This section presents BLF-based adaptive control design that ensures constraint satisfaction and asymptotic output tracking, despite perturbations induced by transient online parameter adaptation. Specifically, we deal with uncertainty in linearly parameterizable nonlinearities

$$f_i(\bar{x}_i) = \theta^T \psi_i(\bar{x}_i), \quad i = 1, \ldots, n \tag{20}$$

where θ, a vector of constant uncertain parameters, belongs to the known compact set Ω_θ, and $\psi \in \mathbb{R}^l$ is a regressor. Let $\hat{\theta}$ be an estimate of θ, $\tilde{\theta} := \hat{\theta} - \theta$, and $\zeta := [y_d, k_{a_1}, k_{b_1}]^T$. Consider the BLF candidate:

$$V = \frac{1}{2p} \log \frac{1}{1 - \xi^{2p}} + \sum_{i=2}^{n} \frac{1}{2} z_i^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}. \tag{21}$$

The control is designed, based on adaptive backstepping with tuning functions (Krstic, Kanellakopoulos, & Kokotovic, 1995), as follows:

$$\alpha_1 = \frac{1}{g_1} \left(-\hat{\theta}^T \omega_1 - (\kappa_1 + \bar{\kappa}_1(t)) z_1 + \dot{y}_d \right)$$

$$\alpha_2 = \frac{1}{g_2} \left(-\hat{\theta}^T \omega_2 - \kappa_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} g_1 x_2 + \sum_{j=0}^{1} \frac{\partial \alpha_1}{\partial \zeta^{(j)}} \zeta^{(j+1)} + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2 - \mu_1 g_1 z_1^{2p-1} \right)$$

$$\alpha_i = \frac{1}{g_i} \left(-\hat{\theta}^T \omega_i - \kappa_i z_i - g_{i-1} z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} g_j x_{j+1} + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \zeta^{(j)}} \zeta^{(j+1)} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma \omega_i z_j \right)$$

$$\omega_1 = \psi_1, \qquad \tau_1 = \mu_1 \omega_1 z_1^{2p-1}$$

$$\omega_i = \psi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \psi_j, \quad i = 2, \ldots, n$$

$$\tau_i = \tau_{i-1} + \omega_i z_i, \quad i = 2, \ldots, n$$

$$u = \alpha_n \qquad \dot{\hat{\theta}} = \Gamma \tau_n. \tag{22}$$

where $\bar{\kappa}_1(t)$ is defined in (12). This yields

$$\dot{V} \leq -\frac{\kappa_1 \xi^{2p}}{1 - \xi^{2p}} - \sum_{i=2}^{n} \kappa_i z_i^2. \tag{23}$$

Theorem 2. *Consider the plant (1) with parametric uncertainty (20), under Assumptions 1–3, and adaptive control (22). If the initial output $y(0)$ satisfies $\underline{k}_{c_1}(0) < y(0) < \bar{k}_{c_1}(0)$, then the following properties hold.*

(i) *The error signals $z_i(t)$, $i = 1, 2, \ldots, n$, are bounded by*

$$-\underline{D}_{z_1}(t) \leq z_1(t) \leq \bar{D}_{z_1}(t), \qquad \|z_{2:n}(t)\| \leq \sqrt{2\bar{V}(0)}$$

for all $t > 0$, where

$$\bar{D}_{z_1}(t) = k_{b_1}(t) \left(1 - e^{-2p\bar{V}(0)} \right)^{\frac{1}{2p}}$$

$$\underline{D}_{z_1}(t) = k_{a_1}(t) \left(1 - e^{-2p\bar{V}(0)} \right)^{\frac{1}{2p}}$$

$$\bar{V} = \frac{1}{2p} \log \frac{1}{1 - \xi^{2p}(0)} + \sum_{i=2}^{n} \frac{1}{2} z_i^2(0) + \frac{1}{2} \lambda_{\max}(\Gamma^{-1}) \max_{\theta \in \partial \Omega_\theta} \|\hat{\theta}(0) - \theta\|^2$$

where $\lambda_{\max}(\bullet)$ is the maximum eigenvalue of $(\bullet)$, and $\partial(\bullet)$ the boundary of set $(\bullet)$.

(ii) *The asymmetric time-varying output constraint is never violated, i.e. $\underline{k}_{c_1}(t) < y(t) < \bar{k}_{c_1}(t)$, $\forall t > 0$.*

(iii) *All closed loop signals are bounded.*

(iv) *The error $z(t) \to 0$ as $t \to \infty$.*

Proof. The proof for parts (i)–(iii) are similar to that of Theorem 1. The main difference is that the transient bound for $z(t)$ is not quantified as a function of time that converges to zero, since we are unable to rewrite (23) into the form $\dot{V} \leq -\rho V$ for any $\rho > 0$. Nevertheless, we are able to show that $z(t) \to 0$ as $t \to \infty$ for part (iv). From (23), using the LaSalle–Yoshizawa Theorem, it follows that $\lim_{t\to\infty} (\kappa_1 \xi^{2p}/(1 - \xi^{2p}) + \sum_{i=2}^{n} \kappa_i z_i^2) = 0$. Thus, we conclude that $z(t) \to 0$ as $t \to \infty$. □

Remark 1. The set of feasible initial conditions $\underline{k}_{c_1}(0) < y(0) < \bar{k}_{c_1}(0)$ in Theorems 1 and 2 is maximal in the sense that the output is able to start from anywhere in the initial constrained output space, i.e. $\underline{k}_{c_1}(0) < y(0) < \bar{k}_{c_1}(0)$, and satisfies $\underline{k}_{c_1}(t) < y(t) < \bar{k}_{c_1}(t)$ $\forall t > 0$. In the special case when $\underline{k}_{c_1}$ and $\bar{k}_{c_1}$ are constant, the proposed control renders the set $\underline{k}_{c_1} < y < \bar{k}_{c_1}$ positively invariant.

Remark 2. In some applications, transient error bounds, not output constraints, need to be enforced. This can be accommodated in the design by directly specifying $k_{b_1}(t)$ and $k_{a_1}(t)$, while omitting $\bar{k}_{c_1}(t)$ and $\underline{k}_{c_1}(t)$.

3.3. Initial output outside constraint region

Practical applications may demand that the output start from outside the constraint region. To accommodate this requirement within our control design framework, we augment a new segment of output constraint $y \in (\underline{k}_{c_0}, \bar{k}_{c_0})$, which extends backwards in time from the start of the original constraint $y \in (\bar{k}_{c_1}, \underline{k}_{c_1})$. The new composite constraint is described by $y \in (\underline{k}_c, \bar{k}_c)$, where

$$\bar{k}_c = \begin{cases} \bar{k}_{c_0}(t), & t \in [-t_0, 0) \\ \bar{k}_{c_1}(t), & t \in [0, \infty) \end{cases}, \qquad \underline{k}_c = \begin{cases} \underline{k}_{c_0}(t), & t \in [-t_0, 0) \\ \underline{k}_{c_1}(t), & t \in [0, \infty) \end{cases}$$

and $t_0 > 0$ denotes the duration for the output to enter the constraint region from its initial value. When designing $\underline{k}_{c_0}(t)$ and $\bar{k}_{c_0}(t)$, we need to ensure that $\underline{k}_c(t)$ and $\bar{k}_c(t)$ satisfy the differentiability conditions in Assumption 2, namely $|\bar{k}_c^{(i)}(t)| \leq \bar{K}_{c_i}$ and $|\underline{k}_c^{(i)}(t)| \leq \underline{K}_{c_i}$, $i = 1, \ldots, n$, $\forall t \in [-t_0, \infty)$, where $\bar{K}_{c_i}$ and $\underline{K}_{c_i}$ are positive constants. This ensures that $\underline{k}_{c_0}^{(i)}(0) = \underline{k}_{c_1}^{(i)}(0)$ and $\bar{k}_{c_0}^{(i)}(0) = \bar{k}_{c_1}^{(i)}(0)$, $i = 0, \ldots, n-1$. Furthermore, for initial output $\underline{y}_0 \leq y(-t_0) \leq \bar{y}_0$, we require that

$$\bar{k}_{c_0}(-t_0) > \bar{y}_0, \qquad \underline{k}_{c_0}(-t_0) < \underline{y}_0.$$

Then, starting from initial time $t = -t_0$, the proposed control ensures that the output is bounded within the augmented constraint, i.e. $y(t) \in (\underline{k}_{c_0}(t), \bar{k}_{c_0}(t))$ for $t \in [-t_0, 0)$. Thereafter, the output satisfies the original constraint, i.e. $y(t) \in (\underline{k}_{c_1}(t), \bar{k}_{c_1})(t)$ for $t \in [0, \infty)$.

3.4. Handling bounded disturbances

The proposed control can also be modified to handle bounded disturbances by dominating the disturbances with adaptive estimates of their bounds, similar to the approach in Ren et al. (2010). Consider the plant (1) with disturbances:

$$\dot{x}_i = f_i + g_i x_{i+1} + d_i(t), \quad i = 1, \ldots, n-1$$

$$\dot{x}_n = f_n + g_n u + d_n(t) \tag{24}$$

where $|d_i(t)| \le D_i$ with $D_i > 0, i = 1, \ldots, n$, constant disturbance bounds. We augment the stabilizing functions and input in (14) with compensation terms that contain adaptive estimates $\hat{D}_i$ of the disturbance bounds:

$$\alpha_{1,d} = \alpha_1 - \frac{\hat{D}_1}{g_1}\tanh\left(\frac{\eta_1}{\delta_1}\right)$$
$$\alpha_{i,d} = \alpha_i + \dot{\alpha}_{i-1,d} - \dot{\alpha}_{i-1} - \frac{\hat{D}_i}{g_i}\tanh\left(\frac{\eta_i}{\delta_i}\right), \quad i = 2, \ldots, n$$
$$u_d = \alpha_{n,d}$$
$$\dot{\hat{D}}_i = \gamma_i\left(\eta_i \tanh\left(\frac{\eta_i}{\delta_i}\right) - \sigma\hat{D}_i\right) \tag{25}$$

where $\eta_1 = \mu_1 z_1^{2p-1}$, $\eta_i = z_i$ for $i = 2, \ldots, n$, and $\delta_i, \gamma_i, \sigma$ are positive constants.

Theorem 3. *Consider the plant with bounded disturbances (24), under Assumptions 1–3 and augmented control (25). If $\underline{k}_{c_1}(0) < y(0) < \overline{k}_{c_1}(0)$, then $\underline{k}_{c_1}(t) < y(t) < \overline{k}_{c_1}(t)\ \forall t > 0$. Further, if there exists a positive number t_d such that, for $t \ge t_d$, $d_i(t) \equiv 0$, $i = 1, \ldots, n$, then $\lim_{t\to\infty} z(t) = 0$.*

Proof. Consider $V_d = V + \sum_{i=1}^n \tilde{D}_i^2/2\gamma_i$, where V is defined in (21). It can be shown, using the identity $\eta_i \tanh(\eta_i/\delta_i) - |\eta_i| \le 0.2785\delta_i$, that

$$\dot{V}_d \le -\frac{\kappa_1\xi^{2p}}{1-\xi^{2p}} - \sum_{i=2}^n \kappa_i z_i^2 + \sum_{i=1}^n D_i\left(|\eta_i| - \eta_i\tanh\left(\frac{\eta_i}{\delta_i}\right)\right)$$
$$+ \sum_{i=1}^n \tilde{D}_i\left(\gamma_i^{-1}\dot{\hat{D}}_i - \eta_i\tanh\left(\frac{\eta_i}{\delta_i}\right)\right)$$
$$\le -\rho V_d + c \tag{26}$$

in the set $|\xi| < 1$, where $\rho = \min_i\{2p\kappa_1, 2\kappa_i, \sigma\gamma_i\}$ and $c = \sum_{i=1}^n D_i(\sigma D_i/2 + 0.2785\delta_i)$ are positive constants. Then, from Ren et al. (2010, Lemma 1), we have that $-k_{a_1}(t) < z_1(t) < k_{b_1}(t)\ \forall t > 0$, and we can show that the output remains constrained despite the disturbances, i.e. $\underline{k}_{c_1}(t) < y(t) < \overline{k}_{c_1}(t)\ \forall t > 0$.

Next, since $d_i(t) \equiv 0$ for $t \ge t_d$, we have

$$\dot{V}_d \le -\rho V_d - \sum_{i=1}^n D_i\eta_i\tanh\left(\frac{\eta_i}{\delta_i}\right), \quad t \ge t_d. \tag{27}$$

From the fact that $D_i\eta_i\tanh(\frac{\eta_i}{\delta_i}) \ge 0$, we obtain that $\dot{V}_d \le -\rho V_d$ for $t \ge t_d$. Along the lines of Theorem 1(i), it can be shown that $-\underline{D}_{z_1}(t) \le z_1(t) \le \overline{D}_{z_1}(t)$, and $\|z_{2:n}(t)\| \le D_{z_{2:n}}(t)\ \forall\, t > t_d$, where

$$\overline{D}_{z_1}(t) := k_{b_1}(t)\left(1 - e^{-2pV(t_d)e^{-\rho(t-t_d)}}\right)^{\frac{1}{2p}}$$
$$\underline{D}_{z_1}(t) := k_{a_1}(t)\left(1 - e^{-2pV(t_d)e^{-\rho(t-t_d)}}\right)^{\frac{1}{2p}}$$
$$D_{z_{2:n}}(t) := \sqrt{2V(t_d)e^{-\rho(t-t_d)}}. \tag{28}$$

Since $\overline{D}_{z_1}(t)$, $\underline{D}_{z_1}(t)$, and $D_{z_{2:n}}(t)$ converge to 0, we conclude that $\lim_{t\to\infty} z(t) = 0$. □

4. Simulation

We present a simulation study to illustrate the performance of the proposed control. Consider the system:

$$\dot{x}_1 = 0.1x_1^2 + x_2$$
$$\dot{x}_2 = 0.1x_1x_2 - 0.2x_1 + (1 + x_1^2)u$$

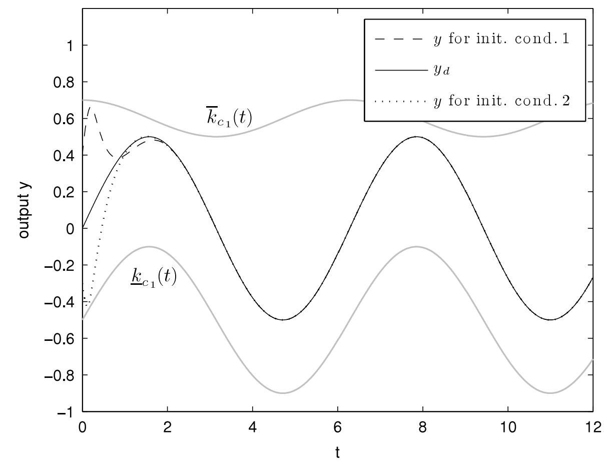

Fig. 2. The output y corresponding to two representative initial points.

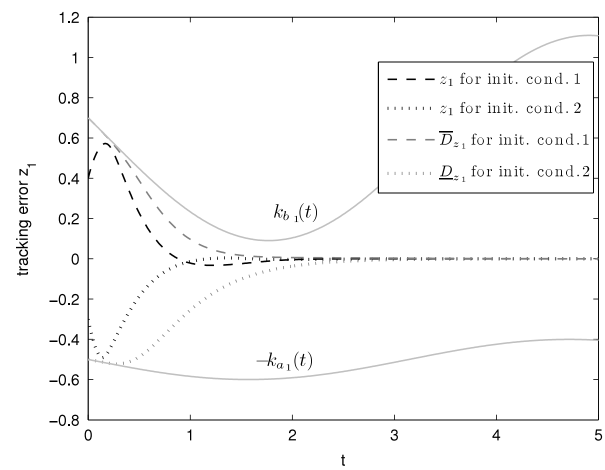

Fig. 3. The tracking error z_1 corresponding to two representative initial points.

with output $y = x_1$. The objective is for $y(t)$ to track a desired trajectory $y_d(t) = 0.5\sin t$ subject to asymmetric output constraints $\overline{k}_{c_1}(t) = 0.6 + 0.1\cos t$ and $\underline{k}_{c_1}(t) = -0.5 + 0.4\sin t$. We apply the control (14) with design parameters $\kappa_1 = \kappa_2 = 2$ and $\beta = 0.1$. Consider two representative initial points, $x(0) = (0.4, 2.5)$ and $x(0) = (-0.3, -2)$, which we annotate as *initial conditions 1 and 2*, respectively. Fig. 2 shows that the output trajectories always satisfy the asymmetric constraint $\underline{k}_{c_1}(t) < y(t) < \overline{k}_{c_1}(t)$ for all $t > 0$, and converge to the desired trajectory $y_d(t)$. Fig. 3 shows that the tracking error trajectories $z_1(t)$ are initially repelled from the bounds $k_{b_1}(t)$ and $-k_{a_1}(t)$, but eventually converge to 0. Indeed, we observe that $z_1(t)$ for *initial condition 1* is upper-bounded by the performance bound $\overline{D}_{z_1}(t)$ and $z_1(t)$ for *initial condition 2* is lower-bounded by $-\underline{D}_{z_1}(t)$. For ease of illustration, we omit $-\underline{D}_{z_1}(t)$ for *initial condition 1* and $\overline{D}_{z_1}(t)$ for *initial condition 2*, since the actual trajectories do not approach these bounds.

5. Conclusions

We have presented a control for strict feedback nonlinear systems with asymmetric time-varying output constraints. We have employed an asymmetric time-varying BLF to prevent transgression of the output constraint, and shown that the output is able to start from anywhere in the initial constrained output space. Asymptotic output tracking has been achieved, and transient performance bound has been quantified as a function of time that converges to zero. An adaptive controller, which ensures constraint satisfaction during the transient phase of online parameter adaptation, has also been proposed. The results in this paper are not only applicable to a larger class of constraint problems, but also improves the performance and enlarges the set of admissible initial outputs when applied to static output constraints.

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