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Thermodynamics-Based Control of Network Systems

The zeroth and first laws of thermodynamics define the concepts of thermal equilibrium and thermal energy. The second law of thermodynamics determines whether a particular transfer of thermal energy can occur. Collectively, these fundamental laws of nature imply that a closed collection of thermodynamic subsystems will tend to thermal equilibrium. This paper generalizes the concepts of energy, entropy, and temperature to undirected and directed networks of single integrators, and demonstrates how thermodynamic principles can be applied to the design of distributed consensus control algorithms for networked dynamical systems. [DOI: 10.1115/1.4023845]

1 Introduction

For a network of interconnected dynamical systems it is often desired that some property of each subsystem approaches a single common value across the network. For example, in a group of autonomous vehicles this property might be a common heading angle or a shared communication frequency. Designing a controller that ensures that a common value will be found is called the *consensus control problem* [1]. Achieving consensus with distributed controllers that can access only local information is called the *distributed consensus control problem*. Related topics include rendezvous, synchronization, flocking, and cyclic pursuit [1]. These topics arise in a broad variety of important applications, including cooperative control of unmanned air vehicles, microsatellite clusters, mobile robots, and congestion control in communication networks.

A sizable body of work has emerged in recent years that address the distributed consensus problem using the tools of algebraic graph theory [1–6]. This paper presents an alternative perspective to the distributed consensus problem, based on *system thermodynamics*, a framework that unifies the foundational disciplines of thermodynamics and dynamical system theory. System thermodynamics has been applied to achieve the formulation of classical thermodynamics in a dynamical systems setting [7]. System thermodynamics has also been used to apply thermodynamic principles to the analysis, design, and control of dynamic systems [8–11]. This paper further examines and extends the latter approach.

To illustrate the relevance of thermodynamics to consensus control, consider conductive heat flow in a homogeneous, isotropic, thermally insulated body. Heat flows within the body according to Fourier's law of heat conduction, which states that the rate of heat flow q through an area A is proportional to the temperature gradient ∇T. The constant of proportionality is an intrinsic material property called the *thermal conductivity* and denoted by κ. Hence, $q = -\kappa A \nabla T$. One consequence of the second law of thermodynamics is that κ cannot be negative; another is that heat will flow until the temperature of the body is uniform. If the system is perturbed by local addition or removal of heat, then the temperature distribution will respond by stabilizing at a new uniform value. That is, the natural flow of heat under Fourier's law robustly maintains a globally stable "temperature consensus" in response to system disturbances and system uncertainty. Just as intuitive notions of energy and dissipation can guide controller design using Lyapunov or passivity-based methods [12], so can the laws of thermodynamics be abstracted and generalized to guide controller design for consensus control problems in networked systems [13].

This paper develops a system thermodynamic framework for the distributed consensus control problem on static, finite-dimensional, undirected and directed networks of first-order systems. This is a somewhat restricted class of problems, which has received extensive attention using other methods. However, system thermodynamics proves extremely effective in designing controllers for such systems, and the intuition provided by the thermodynamic analogies points the way towards extensions to a broader class of problems. The main objective of this paper is to further develop the connection between fundamental thermodynamic principles and the control of network systems.

The contents of this paper are as follows. Section 2 analyzes the dynamics of heat flow in a network of lumped thermal masses from a thermodynamic perspective. Specifically, Sec. 2.1 defines

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a simplified set of thermodynamic concepts and corresponding statements of the zeroth, first, and second laws of thermodynamics. Section 2.2 describes a class of conductive heat transfer mechanisms based on Fourier's law, and shows that this class of mechanisms is consistent with the first and second laws of thermodynamics when applied to heat transfer between a pair of lumped thermal masses. Section 2.3 briefly reviews background concepts from graph theory. Finally, Sec. 2.4 reviews stability theory for systems with nonisolated equilibrium points, and applies this theory to Fourier-type heat transfer in a network of lumped thermal masses.

Section 3 extends the results on thermodynamic systems to abstract dynamical systems. Specifically, Sec. 3.1 presents generalized notions of energy, temperature, and entropy, and obtains a thermodynamics-based stability result for an undirected network. Section 3.2 illustrates the method with a simple simulation example. Section 4 presents a system thermodynamic approach to distributed consensus control on a directed network. Specifically, Sec. 4.1 shows how the subsystem controllers from the undirected case can be modified on a directed network to obtain consensus with conservation of the generalized system energy. This result requires knowledge of the network topology. Section 4.2 illustrates the method with simulation examples. Finally, Sec. 5 summarizes our results and discusses future research directions.

2 Energy, Entropy, and Thermal Equilibria

In this section, we present definitions of internal energy, entropy, and temperature suitable for the analysis of a restricted class of thermodynamic systems. Specifically, we consider a network of n lumped thermal masses interconnected by links along which heat can flow from one subsystem to another. We refer to an individual subsystem as $\mathscr{X}_i$, or simply to subsystem i, and denote the composite system by $\mathscr{X} = \cup_{i=1}^{n} \mathscr{X}_i$.

2.1 The Laws of Thermodynamics. *Internal energy*, or simply *energy*, may be thought of as the potential of a system to perform work on other systems and on the environment. Energy may be transferred between systems by mass transfer, heat transfer, or work. Subsequently, in this paper, energy is transferred only as heat. We denote the internal energy of subsystem i by U_i, and the vector of subsystem energies by $\mathbf{U} = [U_1, U_2, \ldots, U_n]^{\mathrm{T}}$. The total internal energy of the system is the sum of the subsystem energies and is given by

$$U_{\mathscr{X}} = \sum_{i=1}^{n} U_i$$

An *isolated* system does not interchange energy with any other system or the environment. However, energy can be exchanged between the subsystems comprising an isolated system.

Entropy is a measure of how well energy is distributed throughout a system. We denote the entropies of subsystem i by S_i, and the vector of subsystem entropies by $\mathbf{S} = [S_1, S_2, \ldots, S_n]^{\mathrm{T}}$. The system entropy is the sum of the subsystem entropies and is given by

$$S_{\mathscr{X}} = \sum_{i=1}^{n} S_i$$

A higher system entropy indicates a more uniform distribution of energy among subsystems; the subsystem entropies themselves have no meaning in isolation. For a fixed amount of total energy, system entropy is maximal when the energy is *equipartitioned*, that is, when every energy storage mode has equal energy. For details, see Ref. [7].

In classical thermodynamics, *temperature* is well defined only for a system or subsystem in equilibrium. Subsystem i is assumed to be in internal thermal equilibrium, with corresponding temperature $T_i \geq 0$. We denote the vector of subsystem temperatures by $\mathbf{T} = [T_1, T_2, \ldots, T_n]^{\mathrm{T}}$. Any addition or removal of heat is assumed to be sufficiently slow so that each subsystem remains in internal equilibrium. Since subsystems are generally not in equilibrium with each other, there is generally no meaning of "system temperature." When the subsystems are all at a single common temperature $\bar{T}$, then we say the system is in thermal equilibrium with temperature $\bar{T}$. We write $\overline{\mathbf{T}}$ for the temperature vector of a system in thermal equilibrium, that is, $\overline{\mathbf{T}} = \mathbf{e}\bar{T}$, where $\mathbf{e} \triangleq [1, 1, \ldots, 1]^{\mathrm{T}}$ is the n-dimensional ones vector. Here, we distinguish the notion of *thermal equilibrium*, meaning the condition of an isolated system at a uniform temperature, from the notion of *dynamic equilibrium*, meaning the condition of a dynamic system with time rate of change equal to zero.

Energy, entropy, and temperature are related by the *fundamental thermodynamic relationship* [14,15]. In the absence of mechanical work, this relationship can be written as [7]

$$T_i = \left(\frac{\partial S_i}{\partial U_i}\right)^{-1} \tag{1}$$

We will define entropy as a function of energy, that is, $S_i(U_i)$ and $S_{\mathscr{X}}(\mathbf{U})$. It will also be convenient to write the entropies as functions of temperature, using $U_i(T_i)$. Thus, we define $\mathbf{U}(\mathbf{T}) \triangleq [U_1(T_1), U_2(T_2), \ldots, U_n(T_n)]^{\mathrm{T}}$, $\tilde{S}_i(T_i) \triangleq S_i(U_i(T_i))$, and $\tilde{S}_{\mathscr{X}}(\mathbf{T}) \triangleq S_{\mathscr{X}}(\mathbf{U}(\mathbf{T})) = \sum_{i=1}^{n} \tilde{S}_i(T_i)$.

Let $\mathrm{d}Q_i$ be an infinitesimal amount of heat received by subsystem i. Since energy is assumed to be transferred only as heat, $\mathrm{d}U_i = \mathrm{d}Q_i$. The infinitesimal change in entropy that accompanies this heat addition is

$$\mathrm{d}S_i = \left(\frac{\partial S_i}{\partial U_i}\right)\mathrm{d}U_i = \frac{\mathrm{d}Q_i}{T_i} \tag{2}$$

where subsystem $\mathscr{X}_i$ is in equilibrium at temperature T_i. In terms of heat flow rates, Eq. (2) can be written as

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = \frac{1}{T_i}\frac{\mathrm{d}Q_i}{\mathrm{d}t} = \frac{q_i}{T_i} \tag{3}$$

where $q_i \triangleq \dot{Q}_i$. Here, the time rate of change is assumed to be sufficiently slow that the system remains in a slowly varying state of equilibrium. This state is sometimes referred to as *quasi-equilibrium*.

In this paper, we use the following versions of the zeroth, first, and second laws of thermodynamics

2.1.1 Zeroth Law of Thermodynamics (ZLT). If two subsystems are individually in thermal equilibrium with a third subsystem, then the two subsystems are also in thermal equilibrium with each other.

2.1.2 First Law of Thermodynamics (FLT). The increase in the internal energy of a subsystem is equal to the heat supplied to the subsystem.[2] The internal energy of an isolated system is constant.

2.1.3 Second Law of Thermodynamics (SLT). The entropy of an isolated system does not decrease.

2.2 Heat Transfer Between Pairs of Subsystems. We begin by considering heat transfer between a pair of subsystems. Let $\mathrm{d}Q_{ij}$ denote the heat transferred from subsystem j to subsystem i, and let $q_{ij} \triangleq \dot{Q}_{ij}$ denote the rate of flow of heat from subsystem j to subsystem i. We denote the corresponding change (respectively, rate of change) in internal energy and entropy as $\mathrm{d}U_{ij}$ and

[2]This version of the first law holds only when work performed by the system on the environment, and work done by the environment on the system, are assumed to be zero.

$\mathrm{d}S_{ij}$ (respectively, $\dot{U}_{ij}$ and $\dot{S}_{ij}$). It is intuitive to think of two subsystems being linked pairwise by a link that conducts heat, but does not store it. This notion is equivalent to pairwise energy conservation, that is, $q_{ji} = -q_{ij}$. If each subsystem is in quasi-equilibrium, then the SLT implies:

$$\dot{S}_{\mathscr{X}} = \sum_{i=1}^{n} \dot{S}_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \dot{S}_{ij} \geq 0$$

Consider first the case where heat transfer is restricted to be between a single pair of subsystems, namely, subsystem i and subsystem j. This is the case when the system consists of only two subsystems, or when only two subsystems are physically connected. Then the total entropy change is given by

$$\mathrm{d}S_{\mathscr{X}} = \mathrm{d}S_i + \mathrm{d}S_j = \frac{\mathrm{d}Q_{ij}}{T_i} - \frac{\mathrm{d}Q_{ij}}{T_j} = \left(\frac{T_j - T_i}{T_i T_j}\right)\mathrm{d}Q_{ij}$$

or, equivalently, in terms of time rate of change

$$\dot{S}_{\mathscr{X}} = \left(\frac{T_j - T_i}{T_i T_j}\right) q_{ij}$$

By the SLT, $\mathrm{d}S_{\mathscr{X}} \geq 0$ or $\dot{S}_{\mathscr{X}} \geq 0$, implying $\mathrm{sign}(\mathrm{d}Q_{ij}) = \mathrm{sign}(T_j - T_i)$ or $\mathrm{sign}(q_{ij}) = \mathrm{sign}(T_j - T_i)$, where $\mathrm{sign}(\sigma) \triangleq \sigma/|\sigma|$, $\sigma \neq 0$, and $\mathrm{sign}(0) \triangleq 0$. Thus, for heat exchange between subsystem pairs, the SLT implies that if any heat is transferred, then it must move from the subsystem with the higher temperature to the subsystem with the lower temperature.

Definition 1. *The heat transfer law between a pair of subsystems i and j is of* symmetric Fourier type *if it has the form*

$$q_{ij} = \alpha_{ij}(T_j - T_i) \tag{4}$$

where $\alpha_{ij}(\cdot)$ is a function satisfying (i) the sector bound condition

$$\delta_1 \leq \frac{\alpha_{ij}(\xi)}{\xi} \leq \delta_2, \quad \xi \neq 0 \tag{5}$$

with $\alpha(0) = 0$ and $0 < \delta_1 \leq \delta_2$, and (ii) the pairwise symmetry condition

$$\alpha_{ji}(\xi) = -\alpha_{ij}(-\xi) \tag{6}$$

The linear form of Fourier's law, in which $q_{ij} = k_{ij}(T_j - T_i)$ and $q_{ji} = k_{ij}(T_i - T_j)$, is a heat transfer law of symmetric Fourier type. For the heat transfer law between two subsystems to be of symmetric Fourier type, the subsystems must be connected; the "heat transfer law" for an unconnected pair of subsystems, $q_{ij} = q_{ji} = 0$, violates the first inequality in the sector bound, Eq. (5).

If the heat transfer law between a pair of subsystems i and j is of symmetric Fourier type, then the energy transfer between those subsystems automatically satisfies the FLT and the SLT. To see that the FLT is satisfied, note the rate of change of the total internal energy of the subsystem pair is given by

$$\begin{aligned}\dot{U}_{ij} + \dot{U}_{ji} &= \alpha_{ij}(T_j - T_i) + \alpha_{ji}(T_i - T_j)\\ &= \alpha_{ij}(T_j - T_i) - \alpha_{ij}(T_j - T_i)\\ &= 0\end{aligned} \tag{7}$$

where the second equality follows from the symmetry condition, Eq. (6). To see that the SLT is satisfied, note that the rate of change of the total entropy of the subsystem pair is given by

$$\begin{aligned}\dot{S}_{ij} + \dot{S}_{ji} &= \frac{\alpha_{ij}(T_j - T_i)}{T_i} + \frac{\alpha_{ji}(T_i - T_j)}{T_j}\\ &= \left(\frac{1}{T_i} - \frac{1}{T_j}\right)\alpha_{ij}(T_j - T_i)\\ &= \frac{(T_j - T_i)}{T_i T_j}\alpha_{ij}(T_j - T_i)\\ &\geq 0\end{aligned} \tag{8}$$

where the second equality follows from the symmetry condition, Eq. (6), and the inequality follows from the sector bound condition, Eq. (5). The sector bound condition also implies that equality holds in Eq. (8) if and only if $T_i = T_j$.

If every pair of subsystems in a system is either disconnected or obeys a heat transfer law of symmetric Fourier type, then the FLT and the SLT will be satisfied at the system level. To see that the FLT is satisfied, consider the rate of change of the total internal energy of the system given by

$$\begin{aligned}\dot{U}_{\mathscr{X}} &= \sum_{i=1}^{n}\sum_{j=1}^{n} \dot{U}_{ij}\\ &= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \left(\dot{U}_{ij} + \dot{U}_{ji}\right)\\ &= 0\end{aligned} \tag{9}$$

where the second summand is zero due to Eq. (7) if the subsystem pair (i, j) is connected, and due to the fact that $\dot{U}_{ij} = -\dot{U}_{ji} = 0$ if the subsystem pair (i, j) is disconnected.

To see that the SLT is satisfied, consider the rate of change of the total entropy of the system given by

$$\begin{aligned}\dot{S}_{\mathscr{X}} &= \sum_{i=1}^{n}\sum_{j=1}^{n} \dot{S}_{ij}\\ &= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \left(\dot{S}_{ij} + \dot{S}_{ji}\right)\\ &\geq 0\end{aligned} \tag{10}$$

where the second summand is nonnegative due to Eq. (8) if the subsystem pair (i, j) is connected, and due to the fact that $\dot{S}_{ij} = \dot{S}_{ji} = 0$ if the subsystem pair (i, j) is disconnected. Equality in Eq. (10) holds if and only if every connected pair of subsystems is at the same temperature.

2.3 Interconnection Structure. The interconnection of the thermal subsystems strongly influences the equilibrium properties of the system. For example, if the system consists of two or more disjoint sets of subsystems, then these decoupled components will generally not be in thermal equilibrium. We use the framework of graph theory to describe the interconnection structure of the subsystems [16,17]. Every subsystem is a vertex of a directed graph $\mathscr{G}$, with *vertices* $\mathscr{V}_{\mathscr{G}}$ and *directed edges*, or *arcs*, $\mathscr{E}_{\mathscr{G}}$. The total number of vertices in $\mathscr{G}$ is denoted $n_{\mathscr{G}}$. Arcs are written as ordered pairs (j,i). The arc (j,i) is said to *initiate* at j and *terminate* at i. Nodes j and i are called the *tail* and *head*, respectively, of the arc (j,i). Loops are explicitly forbidden in the graphs considered here, that is, there are no arcs of the form (i,i). The total number of arcs in $\mathscr{G}$ terminating at node i is the *in-degree* of i, denoted $d^-_{\mathscr{G}}(i)$, and the total number of arcs in $\mathscr{G}$ initiating at node i is the *out-degree* of i in $\mathscr{G}$, denoted $d^+_{\mathscr{G}}(i)$. If node j is the tail of an arc that terminates at i we say that j is a *direct predecessor* of i, and if node j is the head of an arc that initiates at i we say that j is a *direct successor* of i in $\mathscr{G}$. We denote the set of all direct predecessors of i in $\mathscr{G}$ by $\mathscr{P}_{\mathscr{G}}(i)$, and we denote the set of all direct successors of i in $\mathscr{G}$ by $\mathscr{S}_{\mathscr{G}}(i)$. That is, $\mathscr{P}_{\mathscr{G}}(i) \triangleq \{j : (j,i) \in \mathscr{E}_{\mathscr{G}}\}$ and $\mathscr{S}_{\mathscr{G}}(i) \triangleq \{j : (i,j) \in \mathscr{E}_{\mathscr{G}}\}$.

A *strong path* in $\mathscr{G}$ is an ordered sequence of arcs from $\mathscr{E}_{\mathscr{G}}$ such that the head of any arc is the tail of the next. A strong path in $\mathscr{G}$ can also be considered as a directed subgraph of $\mathscr{G}$. A *strong cycle* is a strong path that begins and ends at the same vertex. Every strong cycle $\mathscr{C}$ satisfies $d^-_{\mathscr{C}}(i) = d^+_{\mathscr{C}}(i)$ for $i \in \mathscr{V}_C$. Vertices may appear in a strong cycle more than once, that is, $d^-_{\mathscr{C}}(i) = d^+_{\mathscr{C}}(i) \geq 1$ for $i \in \mathscr{V}_C$. If no vertex appears more than once, then the cycle is a *simple strong cycle*; for a simple strong cycle, $d^-_{\mathscr{C}}(i) = d^+_{\mathscr{C}}(i) = 1$ for $i \in \mathscr{V}_C$. A graph is *strongly connected* if a strong path exists from any vertex to any other vertex. A property of a strongly connected graph is that it must contain a strong cycle that passes through every vertex in the graph at least once; we call such a cycle a *complete strong cycle*. A complete strong cycle need not include every edge. A strongly connected graph may contain more than one complete strong cycle. A complete strong cycle that contains each vertex exactly once is called a *simple complete strong cycle*. Every simple complete strong cycle satisfies $d^-_{\mathscr{C}}(i) = d^+_{\mathscr{C}}(i) = 1$ for $i \in \mathscr{V}_G$. The nodes of a simple complete strong cycle $\mathscr{C}$ can be renumbered so that $\mathscr{P}_{\mathscr{C}}(i) = \{i-1\}$ and $\mathscr{S}_{\mathscr{C}}(i-1) = \{i\}$ for $i = 1, \ldots, n_{\mathscr{G}}$, where, for notational convenience, node 0 is identified with node $n_{\mathscr{G}}$.

The *adjacency matrix* is $G \triangleq [g_{ij}]$, where $g_{ij} = 1$ if there is an edge initiating at j and terminating at i, that is, if $(j,i) \in \mathscr{E}_{\mathscr{G}}$. Otherwise, $g_{ij} = 0$. By assumption, $[g_{ii}] = 0$. Though all edges are directed, we say that the system graph is *undirected* if $g_{ji} = g_{ij}$, that is, if G is symmetric. Otherwise, the system graph is said to be *directed*.

Remark 1. Heat transfer on an undirected graph can be considered pairwise on edges (i,j) and (j,i). Thus, the FLT and the SLT are automatically satisfied for a heat transfer law of symmetric Fourier type on an undirected graph. A linear heat flow law $q_{ij} = k_{ij}(T_j - T_i)$ is associated with weighted adjacency matrix $K = [k_{ij}]$, which we also call the thermal conductance matrix. The symmetry condition implies $K = K^{\mathrm{T}}$ for the linear form of Fourier's law.

2.4 Thermal Equilibria and Semistability. Consider a network of n subsystems, each with thermal mass M_i, temperature T_i, energy U_i, and entropy S_i. Let the interconnection structure be defined by the graph $\mathscr{G}$. The system is in thermal equilibrium if and only if $\mathbf{T} = \overline{\mathbf{T}} = \mathbf{e}\bar{T}$ for every $\bar{T} > 0$. In the terminology of dynamical system theory, every thermal equilibrium is a *nonisolated* equilibrium point, since every thermal equilibrium with a slightly perturbed uniform temperature will also be an equilibrium point in the dynamical systems sense. Thermal systems have the property that, after a small perturbation, the system will return to thermal equilibrium, though typically at a slightly different temperature. This property is desirable for the distributed consensus control problem. In terms of concepts from dynamical systems theory, neither Lyapunov stability nor asymptotic stability capture this behavior. The relevant concept is that of *semistability*. For a detailed discussion of semistability, see Ref. [12]. In this paper, we require the following definitions for an equilibrium point $\bar{x}$ of an autonomous dynamical system $\dot{x} = f(x)$, where $f : \mathscr{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$. We denote the solution to this system with initial condition $x(0) = x_0 \in \mathscr{D}$ by $x(t; x_0)$

Definition 2. *An equilibrium $\bar{x} \in \mathscr{D}$ is* Lyapunov stable *if, for every $\varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ such that $||x_0 - \bar{x}|| < \delta_1(\varepsilon)$ implies $||x(t; x_0) - \bar{x}|| < \varepsilon$ for all $x_0 \in \mathscr{D}$ and $t \geq 0$.*

Definition 3. *An equilibrium $\bar{x} \in \mathscr{D}$ is* semistable *if it is Lyapunov stable, and if there exists $\delta_2 > 0$ such that for all $x_0 \in \mathscr{D}$ satisfying $||x_0 - \bar{x}|| < \delta_2, x(t; x_0)$ converges to a Lyapunov stable equilibrium point in $\mathscr{D}$, which need not be $\bar{x}$. A set of equilibrium points is* semistable *if every point in the set is semistable.*

Definition 4. *An equilibrium $\bar{x} \in \mathscr{D}$ is* asymptotically stable *if it is Lyapunov stable, and if there exists $\delta_3 > 0$ such that $||x_0 - \bar{x}|| < \delta_3$ implies $x(t; x_0) \to \bar{x}$ as $t \to \infty$. An equilibrium point $\bar{x}$ is* asymptotically stable on a set $\mathscr{W} \subset \mathscr{D}$ *if $\bar{x}$ is asymptotically stable and $x_0 \in \mathscr{W}$ implies $x(t; x_0) \to \bar{x}$ as $t \to \infty$.*

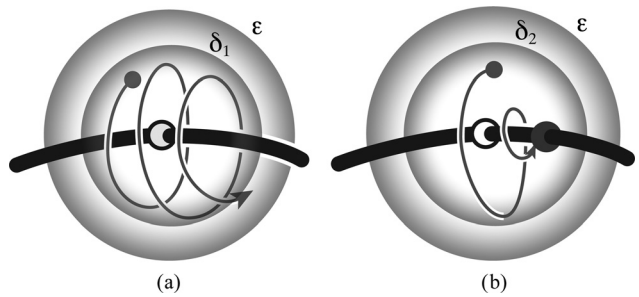

Fig. 1 (*a*) Lyapunov stable nonisolated equilibrium point (hollow). The perturbed trajectory need not converge to a new equilibrium. (*b*) Semistable nonisolated equilibrium point (hollow). Semistability guarantees convergence of the perturbed trajectory to a nearby equilibrium point (filled), and is a stronger property than Lyapunov stability.

Semistability is a stronger property than Lyapunov stability, but a weaker property than asymptotic stability. Nonisolated equilibrium points can be semistable, but not asymptotically stable. For a *set* of nonisolated equilibria, asymptotic stability of the set and semistability of the set are independent properties [12]. Figure 1(*a*) shows a Lyapunov stable nonisolated equilibrium point. A trajectory starting nearby is guaranteed to remain nearby, however it could oscillate forever without converging. Figure 1(*b*) shows a semistable nonisolated equilibrium point. The trajectory converges to a nearby, Lyapunov stable, equilibrium point.

We now proceed to analyze the stability of the thermal equilibria of a thermodynamic system. First, we define the entropy of the ith subsystem by

$$S_i(U_i) \triangleq M_i \ln(U_i) \tag{11}$$

Using Eq. (1), it follows that

$$T_i = \left(\frac{\partial S_i}{\partial U_i}\right)^{-1} = \frac{U_i}{M_i} \tag{12}$$

which gives the familiar equation $U_i = M_i T_i$ relating the energy and temperature of a lumped thermal mass. With this relationship, the subsystem and system entropies can be written directly as a function of temperature, namely,

$$\tilde{S}_i(T_i) \triangleq S_i(M_i T_i) = M_i \ln(M_i T_i), \quad i = 1, \ldots, n$$

and

$$\tilde{S}_{\mathscr{X}}(\mathbf{T}) \triangleq S_{\mathscr{X}}(M_1 T_1, \ldots, M_n T_n) = \sum_{i=1}^{n} \tilde{S}_i(T_i)$$

We refer to $S_{\mathscr{X}}(\mathbf{U})$ or $\tilde{S}_{\mathscr{X}}(\mathbf{T})$ as the *total entropy function*. Other forms of the entropy function may be chosen, as in Refs. [7] and [8], however, the features of those functions will not be required here. One consequence of choosing Eq. (11) is that it does not satisfy the third law of thermodynamics (Nernst's theorem), which states that the entropy is zero when the absolute temperature is zero [7]. This is not a significant drawback for the purposes of this paper.

We first consider isolated systems, for which the total system energy is conserved. The set of feasible subsystem temperatures corresponding to a constant system energy U_0 is given by

$$\mathscr{T}(U_0) \triangleq \left\{\mathbf{T} : T_i \geq 0, \quad i = 1, \ldots, n, \quad \text{and} \quad \sum_{i=1}^{n} M_i T_i = U_0\right\} \tag{13}$$

The entropy function has the following key property for isolated systems.

Theorem 1. *The total entropy function $\tilde{S}_{\mathscr{X}}(\mathbf{T})$, with subsystem entropies $\tilde{S}_i(T_i) = M_i \ln(M_i T_i), i = 1, \ldots, n$, restricted to $\mathscr{T}(U_0)$, has a unique maximum at $\mathbf{T}^* = \mathbf{e}T^*$, where $T^* \triangleq U_0/M$ and $M \triangleq \sum_{i=1}^n M_i$. Equivalently, the total entropy function $S_{\mathscr{X}}(\mathbf{U})$, restricted to the set*

$$\mathscr{U} \triangleq \left\{ \mathbf{U} : U_i \geq 0, \quad i = 1, \ldots, n, \quad \text{and} \quad \sum_{i=1}^n U_i = U_0 \right\}$$

has a unique maximum at $\mathbf{U}^* = [U_1^*, U_2^*, \ldots, U_n^*]^{\mathrm{T}}$, where $U_i^* \triangleq M_i T^*$

Proof. The entropy of the ith subsystem is given by

$$\begin{aligned} \tilde{S}_i(T_i) &= M_i \ln(M_i T_i) \\ &= M_i \ln(M_i T_i) - M_i \ln(M_i T^*) + M_i \ln(M_i T^*) \\ &= M_i \ln(T_i/T^*) + M_i \ln(M_i T^*) \end{aligned}$$

The corresponding total system entropy is given by

$$\tilde{S}_{\mathscr{X}}(\mathbf{T}) = \ln\left(\frac{T_1^{M_1} T_2^{M_2} \cdots T_n^{M_n}}{T^* M_1 + M_2 + \cdots + M_n} \right) + S_{\mathscr{X}}^*$$

where $S_{\mathscr{X}}^* \triangleq \sum_{i=1}^n M_i \ln(M_i T^*)$ is the value of the entropy function at thermal equilibrium. Now, writing

$$\left(\frac{T_1^{M_1} T_2^{M_2} \cdots T_n^{M_n}}{T^{*M_1 + M_2 + \cdots + M_n}} \right) = \left(\frac{T_1^{\mu_1} T_2^{\mu_2} \cdots T_n^{\mu_n}}{T^*} \right)^M$$

where $\mu_i \triangleq M_i/M$, and noting that

$$T^* = U_0/M = (1/M) \sum_{i=1}^n M_i T_i = \sum_{i=1}^n \mu_i T_i$$

it follows that

$$\frac{T_1^{\mu_1} T_2^{\mu_2} \cdots T_n^{\mu_n}}{T^*} = \frac{T_1^{\mu_1} T_2^{\mu_2} \cdots T_n^{\mu_n}}{\mu_1 T_1 + \mu_2 T_2 + \cdots + \mu_n T_n} \leq 1$$

where the inequality follows from the generalized power mean inequality [18], which also provides that equality holds if and only if all the T_i are equal; that is, if and only if $\mathbf{T}$ is of the form $\mathbf{e}T$. However, since

$$T \sum_{i=1}^n M_i = TM = U_0$$

the only $\mathbf{T}$ of this form that satisfies the energy constraint is $\mathbf{e}T^*$. Hence, $\tilde{S}_{\mathscr{X}}(\mathbf{T})$ achieves a maximum value of $S_{\mathscr{X}}^*$ at $\mathbf{T} = \mathbf{T}^*$. The form of the theorem with energy as the independent variable follows from the equality of the two forms of the entropy function $S_{\mathscr{X}}(\mathbf{U}) = \tilde{S}_{\mathscr{X}}(\mathbf{T})$, which implies that $S_{\mathscr{X}}(\mathbf{U})$ must have a unique maximum at $\mathbf{U}^* = [U_1^*, U_2^*, \ldots, U_n^*]^{\mathrm{T}}$, where $U_i^* = M_i T^*$, $i = 1, \ldots, n$. □

Remark 2. One might surmise from Theorem 1 that the maximum entropy does not occur when energy is equipartitioned, since the $\bar{U}_i$'s are not equal, but are instead weighted by the thermal masses M_i. However, energy equipartition refers to uniform distribution of energy over all storage modes, and the subsystems with higher thermal mass have a larger number of storage modes. Maximum entropy corresponds to energy equipartition in this sense.

For positive system energy $U_{\mathscr{X}}$ we denote the temperature vector corresponding to the maximum entropy point by

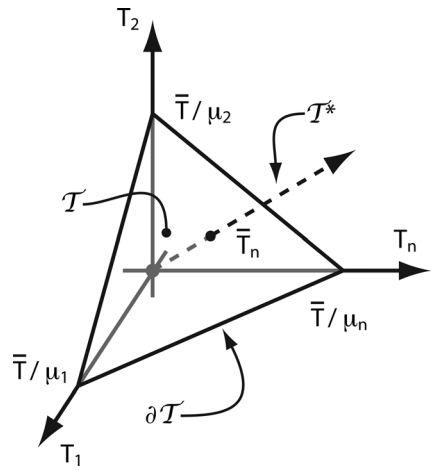

Fig. 2 The subset of feasible temperatures $\mathscr{T}(U_{\mathscr{X}})$ corresponding to total system energy $U_{\mathscr{X}}$

$$\mathbf{T}^*(U_{\mathscr{X}}) \triangleq (U_{\mathscr{X}}/M)\mathbf{e} \tag{14}$$

Figure 2 depicts $\mathscr{T}(U_{\mathscr{X}})$, which is the set of feasible subsystem temperatures corresponding to system energy $U_{\mathscr{X}}$. We denote the set of feasible temperatures for any positive total system energy, which is the union of $\mathscr{T}(U_{\mathscr{X}})$ for all $U_{\mathscr{X}} \geq 0$, by $\mathscr{T}$. Figure 2 also depicts the set $\mathscr{T}^* \triangleq \cup_{U_{\mathscr{X}}>0} \mathbf{T}^*(U_{\mathscr{X}})$, that is, the set of all $\mathbf{T}^*(U_{\mathscr{X}})$ corresponding to a positive total system energy. Finally, the boundary of the feasible set $\mathscr{T}(U_{\mathscr{X}})$ is given by

$$\partial\mathscr{T}(U_{\mathscr{X}}) = \bigcup_{j=1}^n \left\{ \mathbf{T} \in \mathscr{T}(U_{\mathscr{X}}) : T_j = 0 \quad \text{and} \quad \sum_{i=1}^n \mu_i T_i = U_{\mathscr{X}}/M \right\}$$

Since every point on $\partial\mathscr{T}(U_{\mathscr{X}})$ contains at least one $T_i = 0$, it follows that $\lim_{T \to \partial\mathscr{T}} \tilde{S}_{\mathscr{X}}(T) = -\infty$.

The following three properties characterize thermal equilibrium in isolated thermodynamic systems

P1: Total system energy is conserved, that is, $U_{\mathscr{X}}(t) = U_0 \triangleq U_{\mathscr{X}}(0)$

P2: $\mathbf{T}^*(U_0)$ is the unique equilibrium point of the system in $\mathscr{T}(U_0)$. $\mathbf{T}^*(U_0)$ is asymptotically stable in $\mathscr{T}(U_0)$.

P3: Every $\mathbf{T}^*(U_0)$ is a nonisolated equilibrium point in $\mathscr{T}$. The set $\mathscr{T}^*$ is semistable. Every trajectory starting in $\mathscr{T}$ converges to a point in $\mathscr{T}^*$.

Next, we state our main theorem for thermodynamic systems satisfying a heat transfer law of symmetric Fourier-type.

Theorem 2. *Consider a network of interconnected thermal masses with a strongly connected and undirected system graph $\mathscr{G}$. If the energy flow between the subsystems of $\mathscr{G}$ is governed by a heat transfer law of symmetric Fourier type, then properties P1–P3 hold.*

Proof. It was noted in Remark 1 that a heat transfer law of symmetric Fourier type conserves energy if the network is undirected, and hence, P1 holds.

The system of thermal masses with an interconnection law of symmetric Fourier type has dynamics

$$\dot{T}_i = \frac{1}{M_i} \sum_{j \in \mathscr{P}_{\mathscr{G}}(i)} \alpha_{ij}(T_j - T_i), \quad i = 1, \ldots, n \tag{15}$$

To show P2, first note that every $\mathbf{T} = \mathbf{e}T$ is an equilibrium point of Eq. (15). Furthermore, these points are the *only* equilibrium points. To see this, consider any $\mathbf{T}$ not of the form $\mathbf{e}T$, and note that there are a finite number of subsystems. Hence, there must exist at least one subsystem with maximum temperature $T_{\max}$.

Since the network is strongly connected, at least one of the subsystems at $T_{\max}$, say T_M, must have at least one neighbor with a lower temperature. Thus, $\dot{T}_M < 0$, and hence, this system is not in equilibrium. Finally, since $\mathbf{T}^*(U_0) = (U_0/M)\mathbf{e}$ is the only point of the form $\mathbf{e}T$ in $\mathscr{T}(U_0)$, this is the unique equilibrium point.

To show asymptotic stability on $\mathscr{T}(U_0)$, let $\tilde{S}^* = \tilde{S}(\mathbf{T}^*(U_0))$. By Theorem 1, $\tilde{S}^*$ is a unique maximum for the entropy function, and hence, $\tilde{S}(\mathbf{T}) - \tilde{S}^* \leq 0$, with equality holding if and only if $\mathbf{T} = \mathbf{T}^*(U_0)$. The function $V_1(\mathbf{T}) = (1/2)(\tilde{S}(\mathbf{T}) - \tilde{S}^*)^2$ is zero at $\mathbf{T}^*(U_0)$ and positive at every other point in $\mathscr{T}(U_0)$. Furthermore, $\dot{V}_1(\mathbf{T})$ is zero at $\mathbf{T}^*(U_0)$ and negative at every other point in $\mathscr{T}(U_0)$. All that remains to show asymptotic stability of $\mathbf{T}^*(U_0)$ is to shift the origin of $V_1(\mathbf{T})$ to $\mathbf{T}^*(U_0)$ and show that the set $\mathscr{T}(U_0)$ is forward invariant. For the former, it suffices to define $V(\mathbf{T}') \triangleq (1/2)(\tilde{S}(\mathbf{T}' + \mathbf{T}^*(U_0)) - \tilde{S}^*)^2$ and note that $V(\mathbf{T}')$ is positive definite on $\mathscr{T}(U_0)$, and further note that $\dot{V}(\mathbf{T}')$ is negative definite on $\mathscr{T}(U_0)$.

To show forward invariance of the set $\mathscr{T}(U_0)$, consider the portion of the boundary of $\mathscr{T}(U_0)$ corresponding to $T_i = 0$, and take the inner product between the vector $\dot{\mathbf{T}}$ and the unit vector $\hat{\mathbf{u}}_i$ in the direction of T_i. This inner product is equal to the ith component of $\dot{\mathbf{T}}$ evaluated at the boundary point, that is,

$$\langle \dot{\mathbf{T}}, \hat{\mathbf{u}}_i \rangle = \frac{1}{M_i} \sum_{j \in \mathscr{N}_{\mathscr{G}}(i)} \alpha_{ij}(T_j - T_i) \bigg|_{T_i = 0} = \frac{1}{M_i} \sum_{j \in \mathscr{N}_{\mathscr{G}}(i)} \alpha_{ij}(T_j) \tag{16}$$

By the sector bound condition, Eq. (5), $\alpha_{ij}(T_j)/T_j > 0$, and hence, since $T_j \geq 0$ for $j = 1, \ldots, n$, with at least one $T_j > 0$, it follows that $\langle \dot{\mathbf{T}}, \hat{\mathbf{u}}_i \rangle > 0$. This implies that the trajectories of the system point into $\mathscr{T}(U_0)$ along the boundary $\partial\mathscr{T}(U_0)$, and hence, $\mathscr{T}(U_0)$ is an invariant set of Eq. (15). Thus, we conclude [19] that $\mathbf{T}^*(U_0)$ is asymptotically stable in $\mathscr{T}(U_0)$, with region of attraction equal to all of $\mathscr{T}(U_0)$. This completes the proof of P2.

To show the first part of P3, consider the equilibrium point $\mathbf{T}^*(U_0)$ and note that for every τ that does not result in negative temperatures, every point of the form $\mathbf{e}(T^*(U_0) + \tau)$ is also an equilibrium point of Eq. (15). Thus, every neighborhood of $\mathbf{T}^*(U_0)$ in $\mathscr{T}$ contains another equilibrium point, and hence, $\mathbf{T}^*(U_0)$ is nonisolated.

To show that $\mathbf{T}^*(U_0)$ is semistable, we first show that $\mathbf{T}^*(U_0)$ is Lyapunov stable. To do this, consider the open ball of radius δ_1 in $\mathscr{T}$ centered on $\mathbf{T}^*(U_0)$. Consider any initial point $\mathbf{T}'$ in this ball, and denote its energy by U'. Since energy is conserved by a heat transfer law of symmetric Fourier type, the resulting trajectory will converge to equilibrium point $\mathbf{T}^*(U') = \mathbf{e}(U'/M)$. Note that by Eq. (12), U' satisfies $|U' - U_0| \leq \bar{M}\delta_1$, where $\bar{M} \triangleq \max_{i=1,\ldots,n} M_i$. Therefore, $||\mathbf{T}^*(U') - \mathbf{T}^*(U_0)|| = ||\mathbf{e}(U' - U_0)/M|| = \sqrt{n}|U' - U_0|/M \leq \sqrt{n}\delta_1\bar{\mu}$, where $\bar{\mu} = \bar{M}/M$. Given any $\varepsilon > 0$, choose $\delta_1 = \varepsilon/(2\sqrt{n}\bar{\mu})$. Now, since $\mathbf{T}^*(U')$ is asymptotically stable in $\mathscr{T}(U')$, for every $\varepsilon > 0$, choose $\delta_2 > 0$ such that $||\mathbf{T} - \mathbf{T}^*(U')|| < (\varepsilon/2)$ for all $t > 0$. Then, given any $\varepsilon > 0$, choose δ_1 and δ_2 as above, and require that $||\mathbf{T}' - \mathbf{T}(U_0)|| < \min(\delta_1, \delta_2)$, ensuring that $||\mathbf{T}(t) - \mathbf{T}(U_0)|| \leq ||\mathbf{T}(t) - \mathbf{T}(U')|| + ||\mathbf{T}(U') - \mathbf{T}(U_0)|| < \varepsilon$. This shows that $\mathbf{T}^*(U_0)$ is Lyapunov stable. Semistability of $\mathbf{T}^*(U_0)$ now follows immediately from the asymptotic stability of $\mathbf{T}^*(U')$ in $\mathscr{T}(U')$, and the Lyapunov stability of $\mathbf{T}^*(U')$ in $\mathscr{T}$.

Finally, note that, since the system energy is conserved for every trajectory starting in $\mathscr{T}$, for every initial point in $\mathscr{T}$, the system will converge to a point in $\mathscr{T}^*$, which proves that $\mathscr{T}^*$ is semistable. □

3 Thermodynamics of Undirected Networks

By generalizing the notions of temperature, energy, and entropy, system thermodynamics guides the design of distributed consensus controllers that cause networked dynamic systems to emulate natural thermodynamic behavior. On an undirected graph, Theorem 2 can be used with minor modifications to show that every control law of symmetric Fourier type achieves energy consensus. In general, it is information that flows between subsystems, and not energy. This is an important distinction, especially when treating directed networks. In the generalized case, energy flows into or out of a subsystem based on the temperatures of its neighbors, but that energy does not actually come from the neighbors as it would in a natural thermodynamic system.

3.1 Problem Formulation. Consider a network of n single integrators given by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, n \tag{17}$$

where the network has a connectivity structure specified by the *undirected* graph $\mathscr{G}$. Defining the state vector $\mathbf{x} \triangleq [x_1, \ldots, x_n]^{\mathrm{T}}$ and input vector $\mathbf{u} \triangleq [u_1, \ldots, u_n]^{\mathrm{T}}$, the system dynamics, Eq. (17), can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad t \geq 0 \tag{18}$$

This system might be obtained by input-output linearization of a network of relative degree one nonlinear systems. Alternatively, it might arise from a network of feedback passivated nonlinear systems, in which case the subsystems storage functions can serve as energy functions.

The energy of a subsystem depends only on the state of that subsystem. Thus, we define a continuously differentiable, nonnegative internal energy function $U_i(x_i)$ for each subsystem of Eq. (17), and let

$$S_i(U_i) \triangleq M_i \ln(U_i) \tag{19}$$

$$T_i(U_i) \triangleq U_i/M_i \tag{20}$$

$$\tilde{S}_i(T_i) \triangleq M_i \ln(M_i T_i) \tag{21}$$

Furthermore, let

$$U_{\mathscr{X}}(\mathbf{x}) \triangleq \sum_{i=1}^{n} U_i(x_i)$$

$$S_{\mathscr{X}}(\mathbf{U}) \triangleq \sum_{i=1}^{n} S_i(U_i)$$

and

$$\tilde{S}_{\mathscr{X}}(\mathbf{T}) \triangleq \sum_{i=1}^{n} \tilde{S}_i(T_i)$$

To obtain a controller that emulates the desirable stability properties of a thermodynamic system in thermal equilibrium, we specify that the "heat flow" between pairs of subsystems be governed by a law of symmetric Fourier type. Therefore, we take the time derivative of Eq. (20) and require that $\dot{U}_i$ be given by the sum of pairwise heat transfer laws of form Eq. (4). That is, we require that $\dot{U}_i$ be given by

$$\dot{U}_i = \sum_{j \in \mathscr{P}_{\mathscr{G}}(i)} \alpha_{ij}(T_j - T_i), \quad i = 1, \ldots, n \tag{22}$$

where $\alpha_{ij}(\cdot)$ is of symmetric Fourier type. Hence, the evolution of the generalized temperature functions for each subsystem is given by

$$\dot{T}_i = \dot{U}_i/M_i = \frac{1}{M_i} \sum_{j \in \mathscr{P}_{\mathscr{G}}(i)} \alpha_{ij}(T_j - T_i), \quad i = 1, \ldots, n \tag{23}$$

Note that Eq. (23) is identical to Eq. (15), and therefore the generalized subsystem temperatures will behave as physical temperatures in a thermodynamic system. It remains to express the control law, Eq. (22), in terms of the original system variables. Expanding the left-hand side of Eq. (22) gives

$$\dot{U}_i(x_i) = U_i'(x_i)\dot{x}_i = U_i'(x_i)u_i, \quad i = 1, \ldots, n \tag{24}$$

Now, Eq. (22) gives

$$U_i'(x_i)u_i = \sum_{j \in \mathscr{P}_{\mathscr{G}}(i)} \alpha_{ij}(T_j - T_i), \quad i = 1, \ldots, n \tag{25}$$

which can be solved for u_i to obtain the desired feedback control law

$$u_i(\mathbf{x}) = \frac{1}{U_i'(x_i)} \sum_{j \in \mathscr{P}_{\mathscr{G}}(i)} \alpha_{ij}(T_j(U_j(x_j)) - T_i(U_i(x_i))), \quad i = 1, \ldots, n \tag{26}$$

This control is the basis for our main result on undirected graphs.

Theorem 3. *Consider a network of single integrators with a strongly connected, undirected system graph $\mathscr{G}$. For each subsystem of $\mathscr{G}$ define a continuously differentiable, nonnegative generalized energy function $U_i(x_i)$, with corresponding generalized entropy $S_i = M_i \ln(U_i)$ and generalized temperature $T_i = U_i/M_i$. If $U_i'(x_i) \neq 0$ for all x_i, $i = 1, \ldots, n$, then, under the action of every decentralized control law, Eq. (26), where $\alpha_{ij}(\cdot)$ is of symmetric Fourier type, properties P1–P3 hold.*

Proof. The proof is identical to that of Theorem 2 and; hence, is omitted. □

Remark 3. Choosing $U_i(x_i)$ to be strictly increasing for $x_i > 0$ guarantees that $U_i'(x_i) \neq 0$ *for* $x_i > 0, i = 1, \ldots, n$. Note that it is not required that the subsystem energy functions be uniform. As an example for the case where $n = 3$, the mixed set of functions $U_1(x_1) = x_1, U_2(x_2) = e^{x_2}$, and $U_3(x_3) = x_3^2$ is a valid choice for the subsystem energies.

3.2 Illustrative Numerical Examples. Consider a network of four single integrators with the undirected graph structure shown in Fig. 3. We apply a linear control law of symmetric Fourier type to this system for two choices of subsystem energies and temperatures. Specifically, for Case 1 we let $U_i(x_i) = x_i$ and $T_i = U_i$, and for Case 2 we let $U_i(x_i) = (1/2)x_i^2$ and $T_i = U_i$. Note that for both cases, $M_i = 1$ for all i. For the heat transfer law we choose $\alpha_{ij}(\xi) = g_{ij}\xi$ for both cases, where g_{ij} is the (i,j)th entry of the graph connectivity matrix.

For the first simulation,

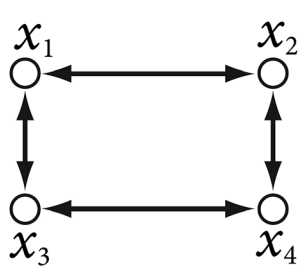

Fig. 3 Simple four-subsystem network

$$u_i(\mathbf{x}) = \sum_{j=1}^{4} g_{ij}(x_j - x_i) \quad \text{for} \quad i = 1, \ldots, 4$$

and

$$\dot{S}_{\mathscr{X}}(\mathbf{x}) = \sum_{i=1}^{4} u_i(\mathbf{x})/x_i$$

This control can be seen to be of the form $\mathbf{u} = -L\mathbf{x}$, where $L = D - K$ is the graph Laplacian and D is the weighted degree matrix. This controller is known to solve the distributed consensus problem on an undirected or balanced graph [2–4]. The response is shown in Fig. 4(*a*)–4(*c*). For the second simulation,

$$u_i(\mathbf{x}) = \sum_{j=1}^{4} -(g_{ij}/2x_i)(x_j^2 - x_i^2) \quad \text{for} \quad i = 1, \ldots, 4$$

and

$$\dot{S}_{\mathscr{X}}(\mathbf{x}) = \sum_{i=1}^{4} (2u_i(\mathbf{x})/x_i)$$

The response is shown in Fig. 4(*d*)–4(*f*).

4 Extensions to Directed Networks

In Sec. 2, we used the property that energy transport is bidirectional, that is, the heat flowing from subsystem j to subsystem i must be equal and opposite to the heat flowing from subsystem i to subsystem j. This is a natural assumption for thermodynamic systems. In Sec. 3, we assumed symmetry of information flow for networked dynamic systems, that is, it was assumed that if temperature T_j is known to subsystem i, then temperature T_i is known to subsystem j. However, many important network consensus problems are posed for *directed* networks, where such symmetry is not guaranteed.

In this section, we once again consider Eq. (17) involving a network of n single integrators. Now, however, the network topology is assumed to be directed. As before, we assume that the graph is strongly connected. From the point of view of the analysis, on a directed graph it is not possible to show compliance with the FLT and the SLT by considering edges pairwise. Instead it is necessary to use the global properties of the graph. The assumption of strong connectedness implies the existence of a complete strong cycle. We construct a distributed control law using this cycle, and show that the result satisfies the FLT and the SLT for the system.

Our controller for a strongly connected directed network will again be based on the laws of heat flow. However, due to the asymmetry of information flow in the directed network, we use an *asymmetric* form of Eq. (4), where the pairwise symmetry condition, Eq. (6), is removed.

Definition 5. *The heat transfer law between a pair of subsystems i and j is of* general Fourier type *if it has the form*

$$q_{ij} = \alpha_{ij}(T_j - T_i) \tag{27}$$

for $j \in \mathscr{P}_{\mathscr{G}}(i), i = 1, \ldots, n$, *where $\alpha_{ij}(\cdot)$ is a function satisfying the sector bound condition*

$$\delta_1 \leq \frac{\alpha_{ij}(\xi)}{\xi} \leq \delta_2, \quad \xi \neq 0 \tag{28}$$

with $\alpha_{ij}(0) = 0$ and $0 < \delta_1 \leq \delta_2$.

4.1 Distributed Consensus Control on a Directed Network With Energy Conservation. To develop a thermodynamics-based consensus controller for the dynamical system, Eq. (17), on

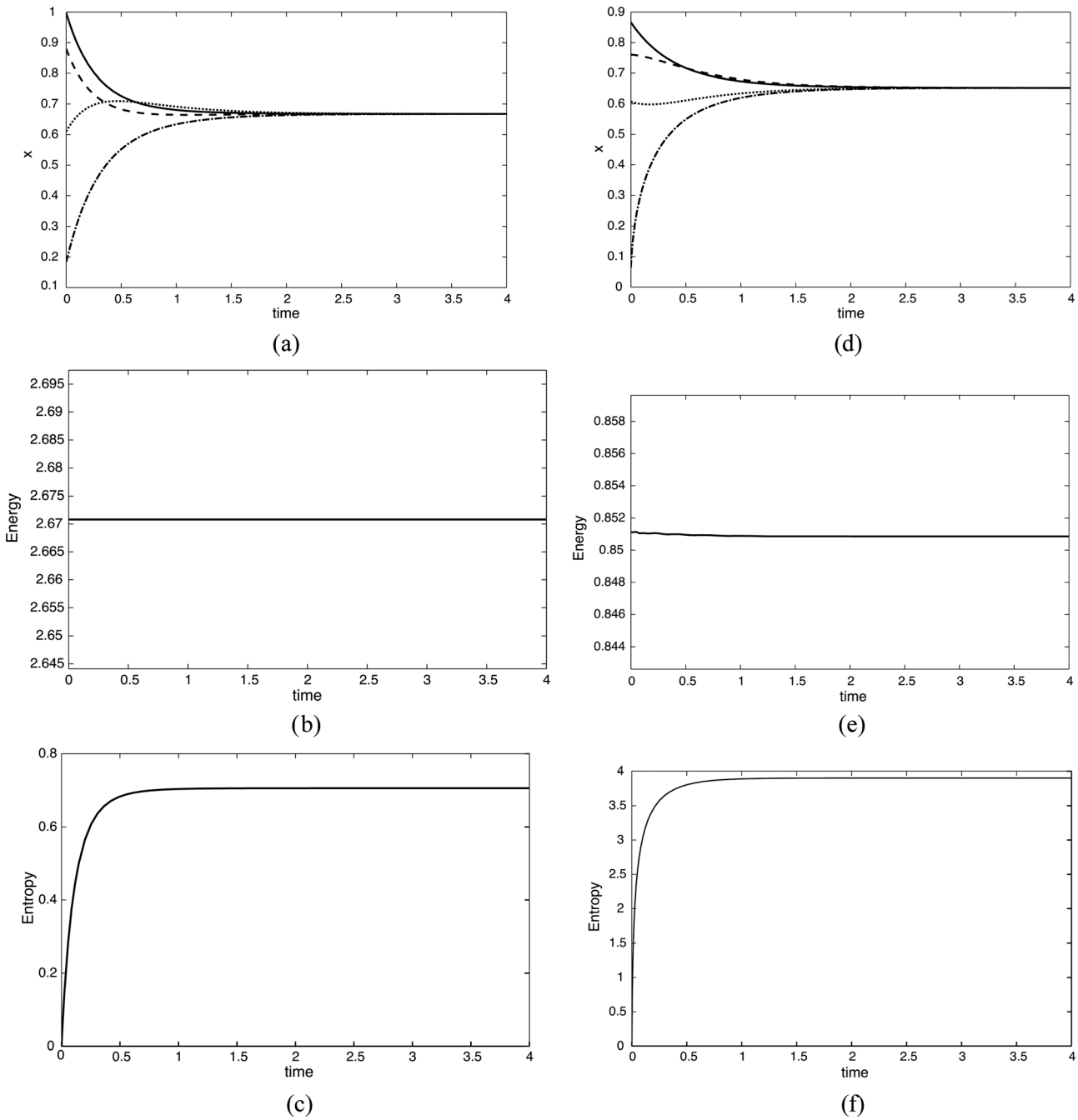

Fig. 4 (*a*)–(*c*) Response of Case 1 showing consensus reached with energy conserved and entropy strictly increasing. (*a*) Consensus variables x_i. (*b*) Energy $U_{\mathscr{X}}$. (c) Entropy $S_{\mathscr{X}}$. (*d*)–(*f*) Response of Case 2 showing consensus reached with energy conserved and entropy strictly increasing. (*d*) Consensus variables $(1/2)x_i^2$. (*e*) Energy $U_{\mathscr{X}}$. (*f*) Entropy $S_{\mathscr{X}}$

a directed network, we let all subsystem thermal masses be equal with value m. Let $U_i(x_i)$ denote the energy of the ith subsystem and let the corresponding subsystem entropy and temperature be given by

$$S_i(U_i) \triangleq m\ln(U_i) \tag{29}$$

$$T_i(U_i) \triangleq U_i/m \tag{30}$$

$$\tilde{S}_i(T_i) \triangleq m\ln(mT_i) \tag{31}$$

The following lemma is needed for our main result on directed networks.

Lemma 1. *For n positive values $T_1, \ldots, T_n$ such that $T_0 = T_n$*

$$\sum_{i=1}^{n} \frac{T_{i-1}}{T_i} \geq n$$

with equality holding if and only if $T_1 = T_2 = \cdots = T_n$.

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(T_1, T_2, \ldots, T_n) \triangleq \frac{T_n}{T_1} + \sum_{i=2}^{n} \frac{T_{i-1}}{T_i} - n \tag{32}$$

and note that $f(T, T, \ldots, T) = 0$ and $f(\lambda T_1, \lambda T_2, \ldots, \lambda T_n) = f(T_1, T_2, \ldots, T_n)$ for every real $\lambda \neq 0$. Since all the T_i's are assumed positive without loss of generality define a reduced set of coordinates by $\mu_i \triangleq T_i/T_n, i = 1, \ldots, n-1$, where every $\mu_i > 0$.

Now, $T_n/T_1 = 1/\mu_1$, $T_{n-1}/T_n = \mu_{n-1}$, and $T_{i-1}/T_i = \mu_{i-1}/\mu_i$, $i = 2, \ldots, n-2$. Therefore, the function Eq. (32) can be written as

$$f(T_1, T_2, \ldots, T_n) = \tilde{f}(\mu_1, \mu_2, \ldots, \mu_{n-1}) = \frac{1}{\mu_1} + \sum_{i=2}^{n-1} \frac{\mu_{i-1}}{\mu_i} + \mu_{n-1} - n$$

Next, note that

$$\frac{\partial \tilde{f}}{\partial \mu_i} = -\frac{\mu_{i-1}}{\mu_i^2} + \frac{1}{\mu_{i+1}}, \quad i = 2, \ldots, n-2 \tag{33}$$

$$\frac{\partial \tilde{f}}{\partial \mu_1} = -\frac{1}{\mu_1^2} + \frac{1}{\mu_2} \tag{34}$$

$$\frac{\partial \tilde{f}}{\partial \mu_{n-1}} = -\frac{\mu_{n-2}}{\mu_{n-1}^2} + 1 \tag{35}$$

Setting Eq. (34) to zero gives $\mu_2 = \mu_1^2$ and setting Eq. (33) to zero gives $\mu_{i+1} = \mu_i^2/\mu_{i-1}$. Proceeding by mathematical induction, we obtain the relationship $\mu_i = \mu_1^i$ for $i = 1, \ldots, n-1$. Setting Eq. (35) to zero yields $\mu_{n-1}^2 = \mu_{n-2}$. Substituting $\mu_i = \mu_1^i$ yields $\mu_1^{2n-2} = \mu_1^{n-2}$, which implies $\mu_1 = 1$. Thus, we obtain $\bar{\mu}_i = 1, i = 1, \ldots, n-1$, where the overbar indicates an extremal value. It follows that $\tilde{f}(\mu_1, \mu_2, \ldots, \mu_{n-1})$ has a unique critical point at $\bar{\mu} = (1, 1, \ldots, 1)$, and; hence, $f(T_1, T_2, \ldots, T_n)$ has a critical point at $T_1 = T_2 = \cdots = T_n$

Furthermore, note that

$$\frac{\partial^2 \tilde{f}}{\partial \mu_i \partial \mu_{i-1}} = -\frac{1}{\mu_i^2}, \quad \frac{\partial^2 \tilde{f}}{\partial \mu_i^2} = -\frac{2\mu_{i-1}}{\mu_i^3}, \quad \frac{\partial^2 \tilde{f}}{\partial \mu_i \partial \mu_{i+1}} = -\frac{1}{\mu_{i+1}^2}, \quad i = 2, \ldots, n-2$$

$$\frac{\partial^2 \tilde{f}}{\partial \mu_1^2} = \frac{2}{\mu_1^3}, \quad \frac{\partial^2 \tilde{f}}{\partial \mu_1 \partial \mu_2} = -\frac{1}{\mu_2^2}$$

$$\frac{\partial^2 \tilde{f}}{\partial \mu_{n-1} \partial \mu_{n-2}} = -\frac{1}{\mu_{n-1}^2}, \quad \frac{\partial^2 \tilde{f}}{\partial \mu_{n-1}^2} = \frac{2\mu_{n-2}}{\mu_{n-1}^3}$$

Evaluating $\left(\partial^2 \tilde{f}/\partial \mu^2\right)$ at $\mu = \bar{\mu}$ yields

$$\left.\frac{\partial^2 \tilde{f}}{\partial \mu^2}\right|_{\mu=\bar{\mu}} = \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix} \tag{36}$$

A tridiagonal symmetric matrix with main diagonal given by $(a_1, a_2, \ldots, a_n)$ and first super- and subdiagonals given by $(b_1, b_2, \ldots, b_{n-1})$ is positive definite if $b_i^2 \leq (1/4)a_i a_{i+1}(1 + (\pi^2/1 + 4\pi^2))$ for $i = 1, \ldots, n-1$ [20]. Therefore, Eq. (36) is positive definite and; hence, the critical point $\bar{\mu} = (1, 1, \ldots, 1)$ is a unique minimum of $\tilde{f}(\mu_1, \mu_2, \ldots, \mu_{n-1})$ in the region where $\mu_i > 0$, $i = 1, \ldots, n-1$. Furthermore, note that

$$\sum_{i=1}^{n} \frac{T_{i-1}}{T_i} - n \geq 0$$

with equality holding if and only if $T_1 = T_2 = \cdots = T_n$. □

The controller is first illustrated for a network that admits a simple complete strong cycle $\mathscr{C}$, that is, a complete strong cycle that passes through every vertex of the graph exactly once. As discussed in Sec. 2.3, the vertices of $\mathscr{C}$ may be numbered so that

$$\mathscr{C} = ((1,2), (2,3), \ldots, (n-1, n), (n, 1))$$

We take the derivative of both sides of Eq. (30) to obtain

$$\dot{T}_i = \frac{1}{m} U_i'(x_i) u_i$$

and let the control for subsystem i be given by

$$u_i(\mathbf{x}) = \frac{\alpha}{U_i'(x_i)} [T_{i-1}(U_{i-1}(x_{i-1})) - T_i(U_i(x_i))], \quad i = 1, \ldots, n \tag{37}$$

where $\alpha > 0$ is now a constant for all connected subsystems, and, for notational convenience, we identify index 0 with subsystem n. The "heat transfer law" underlying this controller is given by

$$\dot{U}_i = q_i = \alpha(T_{i-1} - T_i), \quad i = 1, \ldots, n$$

where α is now a multiplicative positive constant rather than a function. This heat transfer law is of general Fourier type, but not of symmetric Fourier type. Note that in contrast to the controller, Eq. (26), for the undirected graph, which depended on every direct predecessor of i, that is, on every node in $\mathscr{P}_{\mathscr{G}}(i)$, the controller, Eq. (37), depends on *one and only one* direct predecessor of i. While i may have multiple direct predecessors in $\mathscr{G}$, only one is the direct predecessor to i in the complete strong cycle $\mathscr{C}$.

Conservation of total system energy follows immediately from

$$\dot{U}_{\mathscr{X}} = \alpha \sum_{i=1}^{n} (T_{i-1} - T_i) = 0$$

The subsystem entropy functions satisfy $\dot{S}_i = \dot{U}_i/T_i$, that is,

$$\dot{\tilde{S}}_i(T_i) = \frac{\alpha(T_{i-1} - T_i)}{T_i} = \alpha\left(\frac{T_{i-1}}{T_i} - 1\right), \quad i = 1, \ldots, n$$

and

$$\dot{\tilde{S}}_{\mathscr{X}}(\mathbf{T}) = \alpha \sum_{i=1}^{n} \left(\frac{T_{i-1}}{T_i} - 1\right) = \alpha\left[\left(\sum_{i=1}^{n} \frac{T_{i-1}}{T_i}\right) - n\right]$$

It follows from Lemma 1 that $\dot{\tilde{S}}_{\mathscr{X}}(\mathbf{T}) \geq 0$ for all $\mathbf{T} \in \mathscr{T}$, with equality holding if and only if $\mathbf{T} = \bar{\mathbf{T}} = \mathbf{e}\bar{T}$. Note that in general a strongly connected graph can have many distinct simple complete strong cycles. Different choices for $\mathscr{C}$ will result in different controllers, however every such controller will have the properties just derived.

Not all networks admit a simple complete strong cycle. In some cases the shortest complete strong cycle $\mathscr{C}$ may traverse one or more edges, and pass through one or more vertices, multiple times. In such a case, we choose any complete strong cycle $\mathscr{C}$, and let ν_{ji} be the number of times the edge (j, i) is traversed in $\mathscr{C}$. Now, let the distributed control law for subsystem i be given by

$$u_i(\mathbf{x}) = \frac{\alpha}{U_i'(x_i)} \sum_{j \in \mathscr{P}_{\mathscr{C}}(i)} \nu_{ji} \left[T_j(U_j(x_j)) - T_i(U_i(x_i))\right] \tag{38}$$

where $\alpha > 0$ is a constant. This corresponds to heat transfer law

$$\dot{U}_i = q_i = \alpha \sum_{j \in \mathscr{P}_{\mathscr{C}}(i)} \nu_{ji}(T_j - T_i), \quad i = 1, \ldots, n \tag{39}$$

which is of general Fourier type but not of symmetric Fourier type. Next, we state our main result for general directed networks. Note that there can be several distinct complete strong cycles of

$\mathscr{G}$, any of which can be used to construct the distributed control law, Eq. (38).

Theorem 4. *Consider a network $\mathscr{G}$ of strongly connected single integrators. For each subsystem of $\mathscr{G}$ define a continuously differentiable, nonnegative generalized energy function $U_i(x_i)$ and let the subsystem entropies and subsystem temperatures be given by Eqs. (29)– (31). If $U_i'(x_i) \neq 0$ for all $x_i, i = 1, \ldots, n$, then, under the action of the decentralized control, Eq. (38), P1–P3 hold.*

Proof. First, note that the time rate of change of total system energy is given by

$$\begin{aligned}\dot{U}_{\mathscr{X}} &= \sum_{i=1}^{n} \dot{U}_i \\ &= \alpha \sum_{i=1}^{n} \sum_{j \in \mathscr{P}_{\mathscr{C}}(i)} \nu_{ji}(T_j - T_i) \\ &= \alpha \sum_{i=1}^{n} \left(\sum_{j \in \mathscr{S}_{\mathscr{C}}(i)} \nu_{ij} - \sum_{j \in \mathscr{P}_{\mathscr{C}}(i)} \nu_{ji} \right) T_i \\ &= \alpha \sum_{i=1}^{n} \left(d_{\mathscr{C}}^{+}(i) - d_{\mathscr{C}}^{-}(i) \right) T_i \\ &= 0\end{aligned}$$

which shows that $U_{\mathscr{X}}(t) = U_0$, and hence, P1 holds.

To show that the system entropy is nondecreasing under Eq. (38), we again proceed by renumbering the nodes to match their order in the complete strong cycle. This time, however, we do not simply permute the indices, but we also expand the total number of nodes. Specifically, we assign sequential indices to the ordered nodes in the complete strong cycle $\mathscr{C}$, and give a *new index* to each vertex every time it is traversed. Thus, the complete strong cycle $\mathscr{C}$ is converted to a simple complete strong cycle $\mathscr{C}'$ when expressed in the expanded set of indices. The total number of nodes n' in the expanded cycle is given by

$$n' = \sum_{i=1}^{n} \sum_{j \in \mathscr{P}_{\mathscr{C}}(i)} \nu_{ji} \tag{40}$$

where the summation, Eq. (40), gives the number of arcs contained in $\mathscr{C}$, including multiplicity.

Next, the time rate of change of the system entropy is given by

$$\begin{aligned}\dot{\tilde{S}}_{\mathscr{X}}(\mathbf{T}) &= \sum_{i=1}^{n} \dot{\tilde{S}}_i \\ &= \alpha \sum_{i=1}^{n} \sum_{j \in \mathscr{P}_{\mathscr{C}}(i)} \frac{\nu_{ji}}{T_i}(T_j - T_i) \\ &= \alpha \sum_{i=1}^{n} \sum_{j \in \mathscr{P}_{\mathscr{C}}(i)} \nu_{ji} \left(\frac{T_j}{T_i} - 1 \right)\end{aligned} \tag{41}$$

where the summation in Eq. (41) is over each arc in the cycle $\mathscr{C}$, with the factor ν_{ji} accounting for multiplicity. By construction, the summation over all arcs of $\mathscr{C}$ with multiplicity is equal to the summation over all distinct arcs in $\mathscr{C}'$. Denote the vector of subsystem temperatures referenced using the expanded indices as $\mathbf{T}' = [T_1', T_2', T_3', \ldots, T_{n'}']^{\mathrm{T}}$, and denote the representation of the system with these temperatures by $\mathscr{X}'$. Then, we can write

$$\begin{aligned}\dot{\tilde{S}}_{\mathscr{X}}(\mathbf{T}) &= \alpha \sum_{i=1}^{n} \sum_{j \in \mathscr{P}_{\mathscr{C}}(i)} \nu_{ji} \left(\frac{T_j}{T_i} - 1 \right) \\ &= \alpha \sum_{k=1}^{n'} \left(\frac{T_{k-1}'}{T_k'} - 1 \right) \\ &\triangleq \dot{\tilde{S}}_{\mathscr{X}'}(\mathbf{T}')\end{aligned}$$

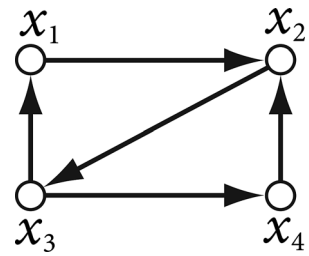

Fig. 5 Directed network of four agents without a simple complete strong cycle

where, for notational convenience, T_0' is identified with $T_{n'}'$. It now follows from Lemma 1 that $\tilde{S}_{\mathscr{X}'}(\mathbf{T}') \geq 0$, where equality holds if and only if $T_1' = T_2' = T_3' = \cdots = T_{n'}'$. From this we conclude that $\tilde{S}_{\mathscr{X}}(\mathbf{T}) \geq 0$, where equality holds if and only if $T_1 = T_2 = T_3 = \cdots = T_n$. The fact that some of the re-indexed temperatures are constrained to be equal implies that the function f defined in the proof of Lemma 1 is constrained to a submanifold in $\mathbb{R}^{n'-1}$. However, this restriction does not change the conclusions of the lemma.

Finally, let $\tilde{S}^* = \tilde{S}(\mathbf{T}^*(U_0))$ and define $V(\mathbf{T}') \triangleq (1/2)(\tilde{S}(\mathbf{T}' + \mathbf{T}^*(U_0)) - \tilde{S}^*)^2$. The rest of the proof is now identical to the proof of Theorem 2 and; hence, is omitted. □

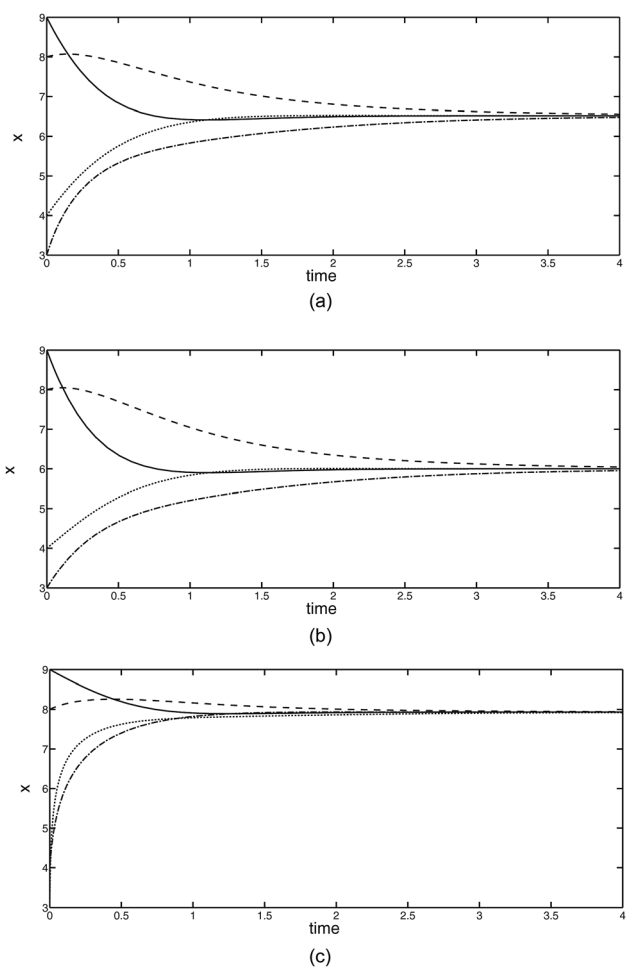

Fig. 6 Simulation of the controlled four agent network for different choices of subsystem energy function. (*a*) $U_i = x_i^2/2$. (*b*) $U_i = x_i$. (*c*) $U_i = e^{x_i}$.

4.2 Illustrative Numerical Example. Consider the four-agent network shown in Fig. 5. The dynamics of each agent are given by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, 4$$

This network does not admit a simple complete strong cycle, however, it does admit the complete strong cycle given by

$$\mathscr{C} = ((1,2),(2,4),(4,3),(3,2),(2,4),(4,1))$$

Now, it follows from Theorem 4 that the controls

$$\begin{aligned} u_1 &= \alpha[\partial U_1/\partial x_1]^{-1}(T_4 - T_1) \\ u_2 &= \alpha[\partial U_2/\partial x_2]^{-1}[(T_1 - T_2) + (T_3 - T_2)] \\ u_3 &= \alpha[\partial U_3/\partial x_3]^{-1}(T_4 - T_3) \\ u_4 &= 2\alpha[\partial U_4/\partial x_4]^{-1}(T_2 - T_4) \end{aligned}$$

achieve network consensus with energy conservation. Figure 6 shows the performance of this controller for three different choices of subsystem energy.

5 Conclusions and Future Work

In this paper, we considered a class of heat transfer mechanisms based on Fourier's law that comply with the first and second laws of thermodynamics. We showed that every heat transfer mechanism from this class will drive interconnected thermal masses to a thermal equilibrium. By generalizing the concepts of internal energy, temperature, and entropy, we showed that decentralized control laws based on these heat transfer laws will guarantee consensus of the generalized temperatures. The formulation has design freedom in the choice of controller and consensus functions. The result was applied and demonstrated on both undirected and directed graphs.

The methods presented here were developed for deterministic, finite-dimensional, input-output linearizable systems with relative degree one. Extending the result to stochastic networks and higher-order subsystems are topics of current research. In addition, the continuous form of Fourier's law ensures robust thermal equilibrium for infinite-dimensional systems. Thus, another natural extension of this work is to extend the approach to infinite dimensional systems.

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