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Robust adaptive control for a class of uncertain strict-feedback nonlinear systems

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SUMMARY

In this paper, robust adaptive control is presented for a class of perturbed strict-feedback nonlinear systems with both completely unknown control coefficients and parametric uncertainties. The proposed design method does not require the *a priori* knowledge of the signs of the unknown control coefficients. For the first time, the key technical Lemma is proven when the Nussbaum function is chosen by $N(\zeta)=\zeta^2\cos(\zeta)$, based on which the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals. Simulation results show the validity of the proposed scheme. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

With the exciting development of adaptive control for parametric uncertain nonlinear systems [1], much attention has been paid to robust adaptive control for nonlinear systems in the presence of unknown disturbances, particularly the class of single-input-single-output (SISO) nonlinear systems that can be transformed in the following strict-feedback form:

$$
\begin{aligned}
\dot{x}_i &= g_i x_{i+1} + \theta_i^{\mathrm{T}} \psi_i(\bar{x}_i) + \Delta_i(t,x), \quad i=1,\ldots,n-1 \\
\dot{x}_n &= g_n u + \theta_n^{\mathrm{T}} \psi_n(x) + \Delta_n(t,x)
\end{aligned}
\tag{1}
$$

where $x=[x_1,\ldots,x_n]^{\mathrm{T}}\in\mathbb{R}^n$, $\bar{x}_i=[x_1,\ldots,x_i]^{\mathrm{T}}$, $i=1,\ldots,n-1$, are the state vectors; $u\in\mathbb{R}$ is the control; $\theta_i\in\mathbb{R}^{n_i}$, $i=1,\ldots,n$, are the unknown constant parameter vectors; n_i's are positive integers;

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$\psi_i(\bar{x}_i)\in\mathbb{R}^{n_i}$, $i=1,\ldots,n$, are known nonlinear functions that are continuous and satisfy $\psi_i(\mathbf{0})=\mathbf{0}$; unknown constants g_i, $i=1,\ldots,n-1$, are referred to as virtual control coefficients [1]; g_n is referred to as the high-frequency gain; and Δ_i's are unknown continuous functions. In system (1), $\theta_i^{\mathrm{T}}\psi_i(\bar{x}_i)$ denotes the parametric uncertainties and $\Delta_i(t,x)$ denotes the non-parametric uncertainties. The unknown nonlinear functions $\Delta_i(t,x)$ could be due to many factors [1, 2] such as measurement noise, modeling errors, external time-varying disturbances, modeling simplifications or changes due to time variations.

In particular when $g_i=1$, robust adaptive control has been developed for system (1) in [2–4] and among others. When g_i's were unknown, several excellent adaptive control algorithms were developed in the literature for nonlinear systems. In [1] where g_i's were assumed to be unknown constants but with known signs, an adaptive control solution was presented for strict-feedback nonlinear systems without disturbances Δ_i's. In [5, 6] where g_i's were functions of states with known signs, adaptive control schemes were proposed for uncertain strict-feedback and pure-feedback nonlinear systems with the aid of neural network parameterization. When g_i's were completely unknown, i.e. with unknown signs, the first solution was given in [7] for a class of first-order linear systems using Nussbaum functions and adaptive control was given in [8] for first-order nonlinear systems. With the help of backstepping design method [1] that has eased the growth conditions on uncertainties, the Nussbaum functions could be used in the controller design for higher-order nonlinear systems in the parameter-strict-feedback form [9–12], or nonlinear systems in the output-feedback form [13–15]. In [7–11, 13, 15], all the closed-loop signals were bounded and the asymptotic regulation or tracking has been achieved.

Compared with the works considering the adaptive control for systems with parametric uncertainties, there were fewer work available in the literature regarding to the robust adaptive control for systems with non-parametric uncertainties, unknown control coefficients and unknown high-frequency gain, among which were References [10, 12, 14–17]. In [10, 12, 14], the disturbances or unknown nonlinear functions were assumed to satisfy certain triangular conditions or bounding conditions, whereas in [15] it was assumed that the disturbances were generated from an external system. In [16, 17], the exponentially decaying terms have been introduced in the controller design to handle the disturbances. The nice properties of Nussbaum functions were difficult to be utilized directly in the stability analysis due to the presence of the exponential terms. In addition, the stability proof had to be function-dependent by fully exploiting the specific Nussbaum functions being chosen.

Motivated by the previous works especially [16, 17], we choose the Nussbaum function as $N(\zeta)=\zeta^2\cos(\zeta)$ in this paper and we find that the proof in [16, 17] cannot be straightforwardly extended and the specific properties of the function need to be investigated fully in the derivation. This is the first time ever that the key technical lemma is proven when the Nussbaum function is chosen by $N(\zeta)=\zeta^2\cos(\zeta)$. The problem of robust adaptive control for the kind of uncertain nonlinear systems can be solvable only after the introduction of the newly introduced technical lemma, and the proposed scheme can be applied to a wide range of systems and problems in robust control domain. We conjecture that the proof of the key technical lemma should be carried out case by case by choosing different Nussbaum functions due to the exponential decaying term.

The main contributions of this paper are as follows: (i) a new technical lemma is introduced, which plays a fundamental role in solving the proposed problem, (ii) the controller does not require *a priori* knowledge of the signs of the unknown control coefficients, and the unknown bounds of the disturbance terms are estimated on-line for improving performance, and (iii) the proposed

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design method expands the class of nonlinear systems for which robust adaptive control approaches have been studied through the introduction of exponential decaying terms in stability analysis.

This paper is organized as follows. The problem formulation and preliminaries are given in Section 2. The design procedures for the robust adaptive control and its main results are presented in Section 3. A simulation example of a voice-coil-motor (VCM) actuator in hard disk drives is given in Section 4 followed by Section 5, which concludes the work. The detailed proof of the key technical lemma is given in Appendix A.

2. PROBLEM FORMULATION AND PRELIMINARIES

The control objective is to construct a robust adaptive nonlinear control law so that the state x_1 of system (1) is driven to a small neighborhood of the origin while keeping all the closed-loop signals bounded. The following assumption is made for the unknown disturbances $\Delta_i(t,x)$, $i=1,\ldots,n$.

Assumption 1
There exist unknown positive constants p_i^*, $1\leqslant i\leqslant n$, such that $\forall(t,x)\in R_+\times R^n$, $|\Delta_i(t,x)|\leqslant p_i^*\phi_i(x_1,\ldots,x_i)$, where ϕ_i is a known non-negative smooth function.

Remark 1
Although the terms $\theta_i^{\mathrm{T}}\psi(\bar{x}_i)$ can be absorbed into $\Delta_i(t,x)$, $i=1,\ldots,n$, for a reduced-order controller, the disadvantage is that the residue error will be large as can be seen from the definitions of μ^*, ρ_i, and c_{i2} later. In addition, for better control performance, knowledge of the system should be fully exploited.

A function $N(\zeta)$ is called a Nussbaum-type function if it has the following properties [7]:

$$\lim_{s\to+\infty}\sup\int_{s_0}^{s}N(\zeta)\,\mathrm{d}\zeta=+\infty \tag{2}$$

$$\lim_{s\to+\infty}\inf\int_{s_0}^{s}N(\zeta)\,\mathrm{d}\zeta=-\infty \tag{3}$$

In comparison with the definition for Nussbaum functions in [9], the definition given by (2) and (3) gives a much larger set of functions, though the example functions satisfy both definitions. Commonly used Nussbaum functions include $\zeta^2\cos(\zeta)$, $\zeta^2\sin(\zeta)$ and $\exp(\zeta^2)\cos((\pi/2)\zeta)$ [18]. For the first time, this paper gives the detailed analysis for the even Nussbaum function, $N(\zeta)=\zeta^2\cos(\zeta)$, $\zeta\in R$.

Lemma 1
Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions defined on $[0,t_f)$ with $V(t)\geqslant 0$, $\forall t\in[0,t_f)$, and $N(\zeta)=\zeta^2\cos(\zeta)$. If the following inequality holds:

$$0\leqslant V(t)\leqslant c_0+\mathrm{e}^{-c_1 t}\int_0^t[g_0N(\zeta)+1]\dot{\zeta}\mathrm{e}^{c_1\tau}\,\mathrm{d}\tau\quad\forall t\in[0,t_f) \tag{4}$$

where constant $c_1>0$, g_0 is a non-zero constant, and $c_0>0$ represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t g_0N(\zeta)\dot{\zeta}\,\mathrm{d}\tau$ are bounded on $[0,t_f)$.

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Proof
See Appendix A. □

Remark 2
Here, we would like to make a conjecture that Lemma 1 is true for all the Nussbaum functions. Because of the presence of $e^{c_1\tau}$ in (4), the proof is function-dependent. We hope that interested reader can prove the lemma for general Nussbaum functions. In addition, we would like to point out that $N(\cdot)$ is not necessarily to be an even function, which is only made for convenience of the proof. If $N(\cdot)$ is chosen as an odd function, e.g. $N(\zeta)=\zeta^2\sin(\zeta)$, the lemma can be easily proven by following the same procedure.

Although the proof is not trivial even for finite t_f, it is the case that $t_f\to\infty$ is of interest. This can be easily extended due to Proposition 1. Consider

$$\dot{x}(t)\in F(x(t)),\quad x(0)=x^0 \tag{5}$$

where $z\mapsto F(z)\subset R^N$ is upper semicontinuous on R^n with non-empty convex and compact values. It is well known that the initial-value problem has a solution and that every solution can be maximally extended.

Proposition 1 (*Ryan* [*19*])
If $x:[0,t_f)\to R^N$ is a bounded maximal solution of (5), then $t_f=\infty$.

3. ROBUST ADAPTIVE CONTROL AND MAIN RESULTS

In this section, the robust adaptive control design procedure for nonlinear system (1) is presented. The design of both the control law and the adaptive law is based on a change of coordinates

$$z_1=x_1$$

$$z_i=x_i-\alpha_{i-1}(x_1,\ldots,x_{i-1},\hat{\theta}_{a,1},\ldots,\hat{\theta}_{a,i-1},\hat{b}_1,\ldots,\hat{b}_{i-1},\zeta_{i-1}),\quad i=1,\ldots,n-1$$

$$z_n=x_n-\alpha_{n-1}(x_1,\ldots,x_{n-1},\hat{\theta}_{a,1},\ldots,\hat{\theta}_{a,n-1},\hat{b}_1,\ldots,\hat{b}_{n-1},\zeta_{n-1})$$

where the functions $\alpha_i, i=1,\ldots,n-1$, are referred to as intermediate control functions, which will be designed using backstepping technique; $\hat{b}_i$ is the parameter estimate for b_i^*, which is the grouped unknown bound for p_i^*; $\hat{\theta}_{a,i}$ represents the estimate of unknown parameter $\theta_{a,i}^*$, which is an augmented parameter and consists of g_j, $j=1,\ldots,i-1$, and θ_j, $j=1,\ldots,i$, as will be clarified later; and ζ_i is the argument of the Nussbaum function. At each intermediate step i, the intermediate control function α_i is designed using an appropriate Lyapunov function V_i and the updating laws $\dot{\hat{b}}_i$, $\dot{\hat{\theta}}_{a,i}$ and $\dot{\zeta}_i$ are given. At the nth step, the actual control u appears and the design is completed.

Step 1: To start, let us consider the subsystem of (1) when $i=1$:

$$\dot{x}_1=g_1x_2+\theta_1^{\mathrm{T}}\psi_1(x_1)+\Delta_1(t,x) \tag{6}$$

Based on the change of coordinates and Assumption 1, the time derivative of $\frac{1}{2}z_1^2$ along (6) is

$$z_1\dot{z}_1=z_1[g_1x_2+\theta_1^{\mathrm{T}}\psi_1(x_1)+\Delta_1(t,x)]\leqslant z_1(g_1x_2+\theta_1^{\mathrm{T}}\psi_1)+b_1^*|z_1|\bar{\phi}_1 \tag{7}$$

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where $b_1^* := p_1^*$, $\bar{\phi}_1 := \phi_1$. For notation consistence, let $\theta_{a,1}^* := \theta_1$ and $\psi_{a,1} := \psi_1$. Consider the Lyapunov function candidate

$$V_1(t) = \tfrac{1}{2}z_1^2 + \tfrac{1}{2}\tilde{\theta}_{a,1}^{\mathrm{T}}\Gamma_1^{-1}\tilde{\theta}_{a,1} + \tfrac{1}{2}\gamma_1^{-1}\tilde{b}_1^2$$

where $\Gamma_1 = \Gamma_1^{\mathrm{T}} > 0$, $\gamma_1 > 0$, $\tilde{\theta}_{a,1} := \hat{\theta}_{a,1} - \theta_{a,1}^*$ and $\tilde{b}_1 := \hat{b}_1 - b_1^*$ denote the estimation error, with $\hat{\theta}_{a,1}$ and $\hat{b}_1$ denoting the parameter estimates of $\theta_{a,1}^*$ and b_1^*, respectively. The time derivative of $V_1(t)$ along (7) is

$$\dot{V}_1 \leqslant z_1(g_1 x_2 + \theta_{a,1}^{*\mathrm{T}}\psi_{a,1}) + b_1^*|x_1|\bar{\phi}_1 + \tilde{\theta}_{a,1}^{\mathrm{T}}\Gamma_1^{-1}\dot{\hat{\theta}}_{a,1} + \gamma_1^{-1}\tilde{b}_1\dot{\hat{b}}_1 \tag{8}$$

Since $x_2 = z_2 + \alpha_1$, consider the following intermediate control and parameter adaptation laws:

$$\alpha_1 = N(\zeta_1)\eta_1 \tag{9}$$

$$\eta_1 = k_1 z_1 + \hat{\theta}_{a,1}^{\mathrm{T}}\psi_{a,1} + \hat{b}_1\bar{\phi}_1 \tanh\left(\frac{z_1\bar{\phi}_1}{\varepsilon_1}\right) \tag{10}$$

$$\dot{\zeta}_1 = z_1\eta_1 \tag{11}$$

$$\dot{\hat{\theta}}_{a,1} = \Gamma_1[z_1\psi_{a,1} - \sigma_{\theta_1}(\hat{\theta}_{a,1} - \theta_{a,1}^0)] \tag{12}$$

$$\dot{\hat{b}}_1 = \gamma_1\left[z_1\bar{\phi}_1 \tanh\left(\frac{z_1\bar{\phi}_1}{\varepsilon_1}\right) - \sigma_{b_1}(\hat{b}_1 - b_1^0)\right] \tag{13}$$

where constant $k_1 := k_{10} + \frac{1}{4} > \frac{1}{4}$, ε_1 is a small positive constant and σ_{θ_1}, σ_{b_1}, $\theta_{a,1}^0$, and b_1^0 are positive design constants. Substituting (9)–(11) into (8) yields

$$\dot{V}_1 \leqslant g_1 z_1 z_2 + g_1 N(\zeta_1)\dot{\zeta}_1 + z_1\theta_{a,1}^{*\mathrm{T}}\psi_{a,1} + b_1^*|x_1|\bar{\phi}_1 + \tilde{\theta}_{a,1}^{\mathrm{T}}\Gamma_1^{-1}\dot{\hat{\theta}}_{a,1} + \gamma_1^{-1}\tilde{b}_1\dot{\hat{b}}_1 \tag{14}$$

Adding and subtracting $\dot{\zeta}_1$ on the right-hand side of (14), and using (12) and (13), we have

$$\begin{aligned}\dot{V}_1 \leqslant & -k_1 z_1^2 + g_1 z_1 z_2 + g_1 N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + b_1^*|x_1|\bar{\phi}_1 - b_1^* x_1\bar{\phi}_1 \tanh\left(\frac{x_1\bar{\phi}_1}{\varepsilon_1}\right)\\ & -\sigma_{\theta_1}(\hat{\theta}_{a,1} - \theta_{a,1}^*)^{\mathrm{T}}(\hat{\theta}_{a,1} - \theta_{a,1}^0) - \sigma_{b_1}(\hat{b}_1 - b_1^*)(\hat{b}_1 - b_1^0)\end{aligned} \tag{15}$$

By completing the squares, i.e.

$$-\sigma_{\theta_1}(\hat{\theta}_{a,1} - \theta_{a,1}^*)^{\mathrm{T}}(\hat{\theta}_{a,1} - \theta_{a,1}^0) \leqslant -\tfrac{1}{2}\sigma_{\theta_1}\|\hat{\theta}_{a,1} - \theta_{a,1}^*\|^2 + \tfrac{1}{2}\sigma_{\theta_1}\|\theta_{a,1}^* - \theta_{a,1}^0\|^2 \tag{16}$$

$$-\sigma_{b_1}(\hat{b}_1 - b_1^*)(\hat{b}_1 - b_i^0) \leqslant -\tfrac{1}{2}\sigma_{b_1}(\hat{b}_1 - b_1^*)^2 + \tfrac{1}{2}\sigma_{b_1}(b_1^* - b_1^0)^2 \tag{17}$$

and using the following nice property [2]:

$$0 \leqslant |x| - x\tanh\left(\frac{x}{\varepsilon}\right) \leqslant 0.2785\varepsilon \quad \text{for } \varepsilon > 0, \quad x \in R \tag{18}$$

Equation (15) becomes

$$\dot{V}_1 \leqslant -k_{10}z_1^2 - \tfrac{1}{2}\sigma_{\theta_1}\|\tilde{\theta}_{a,1}\|^2 - \tfrac{1}{2}\sigma_{b_1}\tilde{b}_1^2 + 0.2785b_1^*\varepsilon_1 + \tfrac{1}{2}\sigma_{\theta_1}\|\theta_{a,1}^* - \theta_{a,1}^0\|^2 + \tfrac{1}{2}\sigma_{b_1}(b_1^* - b_1^0)^2 + g_1N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + g_1^2z_2^2$$

which further leads to

$$\dot{V}_1 \leqslant -c_{11}V_1 + c_{12} + g_1N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + g_1^2z_2^2 \tag{19}$$

where constants $c_{11}, c_{12} > 0$ are defined as

$$c_{11} := \min\left\{2k_{10}, \frac{\sigma_{\theta_1}}{\lambda_{\max}(\Gamma_1^{-1})}, \sigma_{b_1}\gamma_1\right\} \tag{20}$$

$$c_{12} := 0.2785b_1^*\varepsilon_1 + \tfrac{1}{2}\sigma_{\theta_1}\|\theta_{a,1}^* - \theta_{a,1}^0\|^2 + \tfrac{1}{2}\sigma_{b_1}(b_1^* - b_1^0)^2 \tag{21}$$

Let constant $\rho_1 := c_{12}/c_{11} > 0$. Multiplying (19) by $\mathrm{e}^{c_{11}t}$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_1\mathrm{e}^{c_{11}t}) \leqslant c_{12}\mathrm{e}^{c_{11}t} + g_1N(\zeta_1)\dot{\zeta}_1\mathrm{e}^{c_{11}t} + \dot{\zeta}_1\mathrm{e}^{c_{11}t} + g_1^2z_2^2\mathrm{e}^{c_{11}t} \tag{22}$$

Integrating (22) over $[0, t]$, we have

$$0 \leqslant V_1(t) \leqslant \rho_1 + V_1(0) + \mathrm{e}^{-c_{11}t}\int_0^t [g_1N(\zeta_1) + 1]\dot{\zeta}_1\mathrm{e}^{c_{11}\tau}\,\mathrm{d}\tau + \int_0^t g_1^2z_2^2\mathrm{e}^{-c_{11}(t-\tau)}\,\mathrm{d}\tau \tag{23}$$

Remark 3

If there was no uncertain term Δ_1 as in [9, 11], where the uncertainty is from unknown parameters only, adaptive control can be used to solve the problem elegantly and the asymptotic stability can be guaranteed. However, it is not the case here due to the presence of the uncertainty terms Δ_1 in system (1). For illustration, integrating (19) over $[0, t]$ leads to

$$V_1(t) \leqslant V_1(0) + c_{12}t + \int_0^t (g_1N(\zeta_1) + 1)\dot{\zeta}_1\,\mathrm{d}\tau + \int_0^t g_1^2z_2^2\,\mathrm{d}\tau$$

from which no conclusion on the boundedness of $V_1(t)$ or $\zeta_1(t)$ can be drawn by applying Lemma 1 in [9] due to the extra term $c_{12}t$. The problem can be successfully solved by multiplying the exponential term $\mathrm{e}^{c_{11}t}$ to both sides of (19) as in this paper. From (23), the stability results can be drawn by invoking Lemma 1 if $\int_0^t g_1^2z_2^2\mathrm{e}^{-c_{11}(t-\tau)}\,\mathrm{d}\tau$ is upper bounded.

In Equation (23), if there is no extra term $\int_0^t g_1^2z_2^2\mathrm{e}^{-c_{11}(t-\tau)}\,\mathrm{d}\tau$ within the inequality, we can conclude that $V_1(t), \zeta_1, z_1$, and $\hat{\theta}_{a,1}, \hat{b}_1$ are all bounded on $[0, t_f)$ according to Lemma 1. Thus, from Proposition 1, $t_f = \infty$, and we claim that $z_1, \hat{\theta}_{a,1}, \hat{b}_1$ are globally uniformly ultimately bounded. Owing to the presence of term $\int_0^t g_1^2z_2^2\mathrm{e}^{-c_{11}(t-\tau)}\,\mathrm{d}\tau$ in (23), Lemma 1 cannot be applied directly. By noting that

$$\mathrm{e}^{-c_{11}t}\int_0^t g_1^2z_2^2\mathrm{e}^{c_{11}\tau}\,\mathrm{d}\tau \leqslant \mathrm{e}^{-c_{11}t}g_1^2\sup_{\tau\in[0,t]} z_2^2\int_0^t \mathrm{e}^{c_{11}\tau}\,\mathrm{d}\tau \leqslant \frac{g_1^2\sup_{\tau\in[0,t]} z_2^2}{c_{11}}$$

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we know that if z_2 can be regulated as bounded, the boundedness of $\int_0^t g_1^2 z_2^2 \mathrm{e}^{-c_{11}(t-\tau)}\,\mathrm{d}\tau$ is obvious. Then, according to Lemma 1, the boundedness of $z_1(t)$ can be guaranteed. The effect of $\int_0^t g_1^2 z_2^2 \mathrm{e}^{-c_{11}(t-\tau)}\,\mathrm{d}\tau$ will be dealt with in the following steps.

Step i ($2\leqslant i\leqslant n-1$): The similar as in Step 1 procedure is employed recursively for steps from 2 to $n-1$.

The time derivative of $\frac{1}{2}z_i^2$ is

$$z_i\dot{z}_i = z_i\left[g_i x_{i+1}+\theta_i^{\mathrm{T}}\psi_i+\Delta_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(g_j x_{j+1}+\theta_j^{\mathrm{T}}\psi_j+\Delta_j)\right.$$
$$\left.-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j}-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{b}_j}\dot{\hat{b}}_j-\frac{\partial\alpha_{i-1}}{\partial\zeta_{i-1}}\dot{\zeta}_{i-1}\right] \tag{24}$$

Based on Assumption 1, we have

$$z_i\left(\Delta_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\Delta_j\right)\leqslant|z_i|\left(p_i^*\phi_i+\sum_{j=1}^{i-1}\left|\frac{\partial\alpha_{i-1}}{\partial x_j}\right|p_j^*\phi_j\right)\leqslant b_i^*|z_i|\bar{\phi}_i(\bar{x}_i)$$

where $b_i^*:=\max\{p_1^*,\ldots,p_i^*\}>0$ is an unknown constant, and $\bar{\phi}_i(\bar{x}_i)\geqslant\phi_i+\sum_{j=1}^{i-1}|\partial\alpha_{i-1}/\partial x_j|\phi_j>0$ is a known smooth function. A simple example is $\bar{\phi}_i=\phi_i+\sum_{j=1}^{i-1}(\frac{1}{4}(\partial\alpha_{i-1}/\partial x_j)^2+1)\phi_j$.

Therefore, (24) becomes

$$z_i\dot{z}_i\leqslant z_i(g_i x_{i+1}+\theta_{a,i}^{*\mathrm{T}}\psi_{a,i})+b_i^*|z_i|\bar{\phi}_i \tag{25}$$

where

$$\theta_{a,i}^* = [1, g_1,\ldots,g_{i-1},\theta_i^{\mathrm{T}},\theta_1^{\mathrm{T}},\ldots,\theta_{i-1}^{\mathrm{T}}]^{\mathrm{T}}$$
$$\psi_{a,i} = \left[\beta_i,-\frac{\partial\alpha_{i-1}}{\partial x_1}x_2,\ldots,-\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}x_i,\psi_i^{\mathrm{T}},-\frac{\partial\alpha_{i-1}}{\partial x_1}\psi_1^{\mathrm{T}},\ldots,-\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}\psi_{i-1}^{\mathrm{T}}\right]^{\mathrm{T}}$$
$$\beta_i = -\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j}-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{b}_j}\dot{\hat{b}}_j-\frac{\partial\alpha_{i-1}}{\partial\zeta_{i-1}}\dot{\zeta}_{i-1}$$

Consider the Lyapunov function candidate

$$V_i=\tfrac{1}{2}z_i^2+\tfrac{1}{2}\tilde{\theta}_{a,i}^{\mathrm{T}}\Gamma_i^{-1}\tilde{\theta}_{a,i}+\tfrac{1}{2}\gamma_i^{-1}\tilde{b}_i^2$$

where $\Gamma_i=\Gamma_i^{\mathrm{T}}>0$, $\gamma_i>0$, $\tilde{(\cdot)}:=\hat{(\cdot)}-(\cdot)$ denotes the estimation error.

Consider the following adaptive control laws:

$$\alpha_i = N(\zeta_i)\eta_i \tag{26}$$

$$\eta_i = k_i z_i+\hat{\theta}_{a,i}^{\mathrm{T}}\psi_{a,i}+\hat{b}_i\bar{\phi}_i\tanh\left(\frac{z_i\bar{\phi}_i}{\varepsilon_i}\right) \tag{27}$$

$$\dot{\zeta}_i = z_i\eta_i \tag{28}$$

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$$\dot{\hat{\theta}}_{a,i} = \Gamma_i [z_i \psi_{a,i} - \sigma_{\theta_i}(\hat{\theta}_{a,i} - \theta^0_{a,i})] \tag{29}$$

$$\dot{\hat{b}}_i = \gamma_i \left[z_i \bar{\phi}_i \tanh\left(\frac{z_i \bar{\phi}_i}{\varepsilon_i}\right) - \sigma_{b_i}(\hat{b}_i - b^0_i)\right] \tag{30}$$

where constant $k_i := k_{i0} + \frac{1}{4} > \frac{1}{4}$, ε_i is a small positive constant and σ_{θ_i}, σ_{b_i}, $\theta^0_{a,i}$, and b^0_i are positive design constants.

Using (25), the time derivative of $V_i(t)$ along (26)–(30) is

$$\begin{aligned}\dot{V}_i &\leqslant -k_{i0} z_i^2 - \tfrac{1}{2}\sigma_{\theta_i}\|\tilde{\theta}_{a,i}\|^2 - \tfrac{1}{2}\sigma_{b_i}\tilde{b}_i^2 + 0.2785 b_i^* \varepsilon_i + \tfrac{1}{2}\sigma_{\theta_i}\|\theta^*_{a,i} - \theta^0_{a,i}\|^2 + \tfrac{1}{2}\sigma_{b_i}(b^*_i - b^0_i)^2 \\ &\quad + g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + g_i^2 z_{i+1}^2 \\ &\leqslant -c_{i1} V_i + c_{i2} + g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + g_i^2 z_{i+1}^2\end{aligned} \tag{31}$$

where constants $c_{i1}, c_{i2} > 0$ are defined as

$$c_{i1} := \min\left\{2k_{i0}, \frac{\sigma_{\theta_i}}{\lambda_{\max}(\Gamma_i^{-1})}, \sigma_{b_i}\gamma_i\right\} \tag{32}$$

$$c_{i2} := 0.2785 b^*_i \varepsilon_i + \tfrac{1}{2}\sigma_{\theta_i}\|\theta^*_{a,i} - \theta^0_{a,i}\|^2 + \tfrac{1}{2}\sigma_{b_i}(b^*_i - b^0_i)^2 \tag{33}$$

By defining $\rho_i := c_{i2}/c_{i1} > 0$, we can similarly obtain

$$0 \leqslant V_i(t) \leqslant \rho_i + V_i(0) + \mathrm{e}^{-c_{i1}t}\int_0^t [g_i N(\zeta_i) + 1]\dot{\zeta}_i \mathrm{e}^{c_{i1}\tau}\,\mathrm{d}\tau + \int_0^t g_i^2 z_{i+1}^2 \mathrm{e}^{-c_{i1}(t-\tau)}\,\mathrm{d}\tau$$

Remark 4

Similarly, if z_{i+1} can be regulated as bounded, and $\int_0^t g_i^2 z_{i+1}^2 \mathrm{e}^{-c_{i1}(t-\tau)}\,\mathrm{d}\tau$ is therefore bounded at the following steps, then according to Lemma 1, the boundedness of $z_i(t)$ can be guaranteed.

Step n: In the final step, the actual control u appears. Similarly, the time derivative of $\frac{1}{2}z_n^2$ is

$$\begin{aligned} z_n\dot{z}_n &\leqslant z_n\left[g_n u + \theta_n^{\mathrm{T}}\psi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{x_j}(g_j x_{j+1} + \theta_j^{\mathrm{T}}\psi_j) - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{b}_j}\dot{\hat{b}}_j\right] \\ &\quad + b^*_n|z_n|\bar{\phi}_n \\ &= z_n\left[g_n u + \theta_n^{\mathrm{T}}\psi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^{\mathrm{T}}\psi_j) + \beta_n\right] + b^*_n|z_n|\bar{\phi}_n \\ &= z_n[g_n u + \theta^{*T}_{a,n}\psi_{a,n}] + b^*_n|z_n|\bar{\phi}_n \end{aligned} \tag{34}$$

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where

$$b_n^* = \max\{p_1^*, \ldots, p_n^*\}$$

$$\bar{\phi}_n(\bar{x}_n) = \phi_n + \sum_{j=1}^{n-1} \left| \frac{\partial \alpha_{n-1}}{\partial x_j} \right| \phi_j$$

$$\beta_n = -\sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_{a,j}} \dot{\hat{\theta}}_{a,j} - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{b}_j} \dot{\hat{b}}_j$$

$$\theta_{a,n}^* = [1, g_1, \ldots, g_{n-1}, \theta_n^{\mathrm{T}}, \theta_1^{\mathrm{T}}, \ldots, \theta_{n-1}^{\mathrm{T}}]^{\mathrm{T}}$$

$$\psi_{a,n} = \left[\beta_n, -\frac{\partial \alpha_{n-1}}{\partial x_1} x_2, \ldots, -\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} x_n, \psi_n^{\mathrm{T}}, -\frac{\partial \alpha_{n-1}}{\partial x_1} \psi_1^{\mathrm{T}}, \ldots, -\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} \psi_{n-1}^{\mathrm{T}} \right]^{\mathrm{T}}$$

As this is the last step, the final adaptive control laws are given explicitly as follows:

$$u = N(\zeta_n)\eta_n \tag{35}$$

$$\eta_n = k_n z_n + \hat{\theta}_{a,n}^{\mathrm{T}} \psi_{a,n} + \hat{b}_n \bar{\phi}_n \tanh\left(\frac{z_n \bar{\phi}_n}{\varepsilon_n} \right) \tag{36}$$

$$\dot{\zeta}_n = z_n \eta_n \tag{37}$$

$$\dot{\hat{\theta}}_{a,n} = \Gamma_n [z_n \psi_{a,n} - \sigma_{\theta_n} (\hat{\theta}_{a,n} - \theta_{a,n}^0)] \tag{38}$$

$$\dot{\hat{b}}_n = \lambda_n \left[z_n \bar{\phi}_n \tanh\left(\frac{z_n \bar{\phi}_n}{\varepsilon_n} \right) - \sigma_{b_n} (\hat{b}_n - b_n^0) \right] \tag{39}$$

where constant $k_n > 0$ (different from $k_i > \frac{1}{4}$ in the intermediate steps) and ε_n is a small positive constant, $\Gamma_n = \Gamma_n^{\mathrm{T}} > 0$, λ_n, σ_{θ_n}, σ_{b_n}, $\theta_{a,n}^0$ and b_n^0 are positive design constants.

Consider the Lyapunov function candidate

$$V_n(t) = \tfrac{1}{2} z_n^2 + \tfrac{1}{2} \tilde{\theta}_{a,n}^{\mathrm{T}} \Gamma_n^{-1} \tilde{\theta}_{a,n} + \tfrac{1}{2} \gamma_n^{-1} \tilde{b}_n^2$$

The time derivative of V_n satisfies

$$\begin{aligned} \dot{V}_n(t) \leqslant & -k_n z_n^2 - \tfrac{1}{2} \sigma_{\theta_n} \|\tilde{\theta}_{a,n}\|^2 - \tfrac{1}{2} \sigma_{b_n} \tilde{b}_n^2 + 0.2785 b_n^* \varepsilon_n + \tfrac{1}{2} \sigma_{\theta_{a,n}} |\theta_{a,n}^* - \theta_{a,n}^0|^2 \\ & + \tfrac{1}{2} \sigma_{b_n} (b_n^* - b_n^0)^2 + g_n N(\zeta_n) \dot{\zeta}_n + \dot{\zeta}_n \end{aligned} \tag{40}$$

which yields

$$0 \leqslant V_n(t) \leqslant \rho_n + V_n(0) + \mathrm{e}^{-c_{n1} t} \int_0^t [g_n N(\zeta_n) + 1] \dot{\zeta}_n \mathrm{e}^{c_{n1} \tau} \, \mathrm{d}\tau$$

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where constant $\rho_n := c_{n2}/c_{n1} > 0$, constants $c_{n1}, c_{n2} > 0$ are defined as

$$c_{n1} := \min\left\{2k_n, \frac{\sigma_{\theta_n}}{\lambda_{\max}(\Gamma_n^{-1})}, \sigma_{b_n}\lambda_n\right\} \tag{41}$$

$$c_{n2} := 0.2785 b_n^* \varepsilon_n + \tfrac{1}{2}\sigma_{\theta_n}\|\theta_{a,n}^* - \theta_{a,n}^0\|^2 + \tfrac{1}{2}\sigma_{b_n}(b_n^* - b_n^0)^2 \tag{42}$$

Using Lemma 1, we can conclude that $\zeta_n(t)$ and $V_n(t)$, and hence $z_n(t)$, $\hat{\theta}_{a,n}(t)$, $\hat{b}_{a,n}(t)$, are bounded on $[0, t_f)$. From the boundedness of $z_n(t)$, the boundedness of the extra term $\int_0^t g_{n-1}^2 z_n^2 \mathrm{e}^{-c_{n-1,1}(t-\tau)}\,\mathrm{d}\tau$ at Step $(n-1)$ is readily obtained. Applying Lemma 1 backward $(n-1)$ times, it can be seen from the above design procedures that $V_i(t)$, $z_i(t)$, $\hat{\theta}_{a,i}(t)$, $\hat{b}_{a,i}(t)$, and hence $x_i(t)$, are bounded on $[0, t_f)$.

The stability and performance of the closed-loop system under the above adaptive scheme is summarized in Theorem 1.

Theorem 1

Consider the controller (35)–(37) with the parameter updating laws (38) and (39) for the uncertain strict-feedback nonlinear system (1) with completely unknown control coefficients g_i under Assumption 1. The resulting closed-loop system satisfies the following properties under bounded initial conditions:

(i) all the signals are globally uniformly ultimately bounded;

(ii) given any $\mu > \mu^* = \sqrt{\sum_{i=1}^n 2(\rho_i + c_i)}$, there exists T such that, for all $t \geqslant T$, $\|z(t)\| \leqslant \mu$, where $z(t) := [z_1, \ldots, z_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\rho_i := c_{i2}/c_{i1}$, $i = 1, \ldots, n$, and c_i is the upper bound of $\int_0^t [g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + g_i^2 z_{i+1}^2]\mathrm{e}^{-c_{i1}(t-\tau)}\,\mathrm{d}\tau$, $i = 1, \ldots, n-1$, and c_n is the upper bound of $\int_0^t [g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n]\mathrm{e}^{-c_{n1}(t-\tau)}\,\mathrm{d}\tau$, and the compact set $\Omega_z = \{z \in \mathbb{R}^n \mid \|z(t)\| \leqslant \mu\}$ can be adjusted by appropriately choosing the design parameters;

(iii) the state $x_1(t)$ satisfies

$$|x_1(t)| \leqslant \sqrt{2V_1(0)\mathrm{e}^{-c_{11}t} + 2(\rho_1 + c_1)} \quad \forall t \geqslant 0 \tag{43}$$

Proof

(i) Following the design procedures from Step 1 to Step n, it can be obtained that $V_i(t)$, $z_i(t)$, $\zeta_i(t)$, $\hat{\theta}_{a,i}(t)$, $\hat{b}_{a,i}(t)$, and $x_i(t)$ are bounded on $[0, t_f)$ by invoking Lemma 1. According to Proposition 1, if the solution of the closed-loop system is bounded, then $t_f = \infty$. Therefore, all the signals in the closed-loop system are globally uniformly ultimately bounded.

(ii) From (40) at Step n, we have

$$\dot{V}_n(t) \leqslant -c_{n1}V_n(t) + c_{n2} + g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n \tag{44}$$

which yields

$$0 \leqslant V_n(t) \leqslant [V_n(0) - \rho_n]\mathrm{e}^{-c_{n1}(t)} + \rho_n + \int_0^t [g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n]\mathrm{e}^{-c_{n1}(t-\tau)}\,\mathrm{d}\tau \tag{45}$$

with constants $c_{n1}, c_{n2} > 0$ defined in (41) and (42), and constant $\rho_n \triangleq c_{n2}/c_{n1} > 0$.

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Invoking Lemma 1, $\zeta_n(t)$, $z_n(t)$ $V_n(t)$, $\int_0^t g_n N(\zeta_n)\dot{\zeta}_n \,\mathrm{d}\tau$ are bounded. Therefore, $\int_0^t [g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n]\mathrm{e}^{-c_{n1}(t-\tau)}\,\mathrm{d}\tau$ is bounded. Then we can define $c_n \triangleq \sup_{\tau\in[0,t]} |\int_0^t [g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n]\mathrm{e}^{-c_{n1}(t-\tau)}\,\mathrm{d}\tau| > 0$ such that (45) becomes

$$0 \leqslant V_n(t) \leqslant [V_n(0) - \rho_n]\mathrm{e}^{-c_{n1}t} + \rho_n + c_n$$

Since $\frac{1}{2}z_n^2(t) \leqslant V_n(t)$, we have

$$z_n^2(t) \leqslant 2[V_n(0) - \rho_n]\mathrm{e}^{-c_{n1}t} + 2(\rho_n + c_n) \tag{46}$$

As $z_n(t)$ is proved to be bounded, invoking Lemma 1 at Step $(n-1)$, the boundedness of $\zeta_{n-1}(t)$, $z_{n-1}(t)$ $V_{n-1}(t)$, $\int_0^t g_{n-1}N(\zeta_{n-1})\dot{\zeta}_{n-1}\,\mathrm{d}\tau$ can be guaranteed. Therefore, the integral $\int_0^t [g_{n-1}N(\zeta_{n-1})\dot{\zeta}_{n-1} + \dot{\zeta}_{n-1} + g_{n-1}^2 z_n^2]\mathrm{e}^{-c_{(n-1)1}(t-\tau)}\,\mathrm{d}\tau$ is bounded as well with its upper bounded defined as $c_{n-1} > 0$. Applying Lemma 1 backward until Step 1, similarly it can be obtained that

$$z_i^2(t) \leqslant 2[V_i(0) - \rho_i]\mathrm{e}^{-c_{i1}t} + 2(\rho_i + c_i), \quad i = n-1, \ldots, 1 \tag{47}$$

Combining (46) and (47) leads to

$$\|z(t)\| \leqslant \sqrt{\sum_{i=1}^{n} \{2[V_i(0) - \rho_i]\mathrm{e}^{-c_{i1}t} + 2(\rho_i + c_i)\}} \tag{48}$$

Let $\mu^* = \sqrt{\sum_{i=1}^n 2(\rho_i + c_i)}$. From (48), we can conclude that given any $\mu > \mu^*$, there exists T, such that for any $t > T$, $\|z(t)\| \leqslant \mu$ holds. Specifically, for $\mu > \mu^*$ defined by

$$\mu = \sqrt{\sum_{i=1}^{n} \{2[V_i(0) - \rho_i]\mathrm{e}^{-c_{i1}T} + 2(\rho_i + c_i)\}} \tag{49}$$

the corresponding T is given by

$$T = T(\mu, V(0)) = -\ln\left(\frac{\mu^2 - \sum_{i=1}^n 2(\rho_i + c_i)}{\sum_{i=1}^n 2[V_i(0) - \rho_i]}\right) \Bigg/ \sum_{i=1}^{n} c_{i1} \tag{50}$$

(iii) As $x_1(t) = z_1(t)$, (43) can be readily obtained from (47) for $i = 1$.

In addition, by appropriately choosing the design parameters, we can adjust the regulation accuracy of the state $x_1(t)$ while keeping the boundedness of all the signals in the close-loop system. However, trade-off should be made between the transient performance such as overshoot or settling time and the steady-state regulation/tracking accuracy. □

Remark 5

Decreasing ε_i, σ_{θ_i} and σ_{b_i} will help to reduce the size of Ω_z. However, if ε_i, σ_{θ_i} and σ_{b_i} are too small, it may not be enough to prevent the parameter estimates from drifting to very large values in the presence of disturbance, where the large $\hat{\theta}_i$ might result in a variation of a high-gain control. Therefore, in practical applications, the design parameters should be adjusted carefully for achieving suitable transient performance and control action.

Under additional assumptions, regulation of the system state $x = [x_1, \ldots, x_n]^{\mathrm{T}}$ to the origin can be achieved as shown in the following corollary.

Corollary 1
Under the conditions of Theorem 1, if functions ψ_i in system (1) and ϕ_i in Assumption 1 vanish at the origin, then we can find an adaptive controller of the form of (35)–(39) with $\sigma_{\theta_i}=\sigma_{b_i}=0, i=1,\ldots,n$ such that all the solutions of the closed-loop system satisfy $\lim_{t\to\infty}\|x(t)\|=0$.

Proof
Following the same design procedure, in the present case, we have

$$\dot{V}_i \leqslant -k_{i0}z_i^2+g_i N(\zeta_i)\dot{\zeta}_i+\dot{\zeta}_i+g_i^2 z_{i+1}^2, \quad i=1,\ldots,n-1 \tag{51}$$

$$\dot{V}_n \leqslant -k_n z_n^2+g_n N(\zeta_n)\dot{\zeta}_n+\dot{\zeta}_n \tag{52}$$

From (52) and using Lemma 1, it follows that $\zeta_n(t)$ and $V_n(t)$, hence $z_n(t)$, $\hat{\theta}_{a,n}(t)$, $\hat{b}_n(t)$, are globally uniformly ultimately bounded. Moreover, $z_n(t)$ is square integrable. Noting (51) and applying Lemma 1 backward $(n-1)$ times, it can be obtained that $V_i(t)$, $z_i(t)$, $\hat{\theta}_{a,i}(t)$, $\hat{b}_i(t)$, and hence $x_i(t)$, are globally uniformly ultimately bounded. In addition, since $\dot{x}_i$, $1\leqslant i\leqslant n$, are bounded, functions $x_i(t)$ are uniformly continuous. Hence, a direct application of Barbalat's lemma gives that $\lim_{t\to\infty}\|x(t)\|=0$. □

4. SIMULATION STUDIES

Consider the dynamics of a VCM actuator in the following form:

$$\frac{1}{k}\ddot{y}(t)=u(t)-\Delta \tag{53}$$

where y is the head position of the actuator, u is the control signal, and Δ represents the torque disturbances including pivot friction, windage and bias, or any unmodeled dynamics, which satisfies

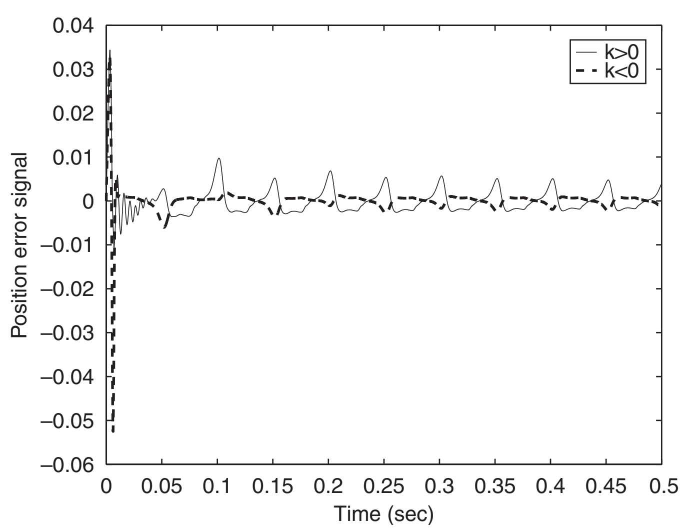

Figure 1. Position error signal.

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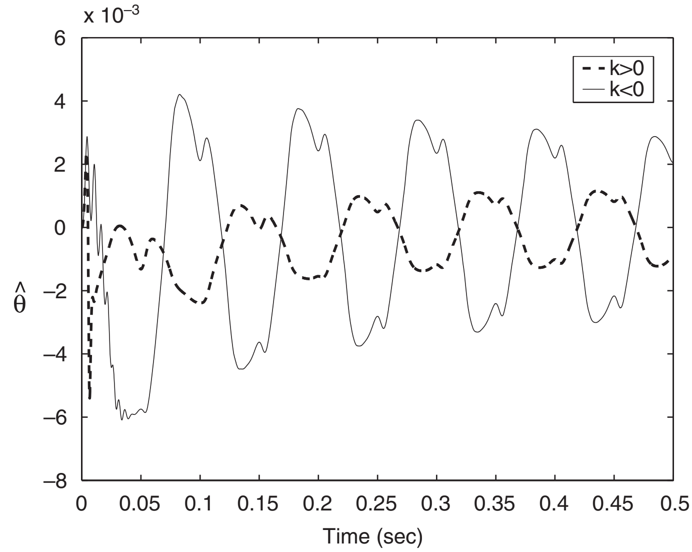

Figure 2. Parameter estimate.

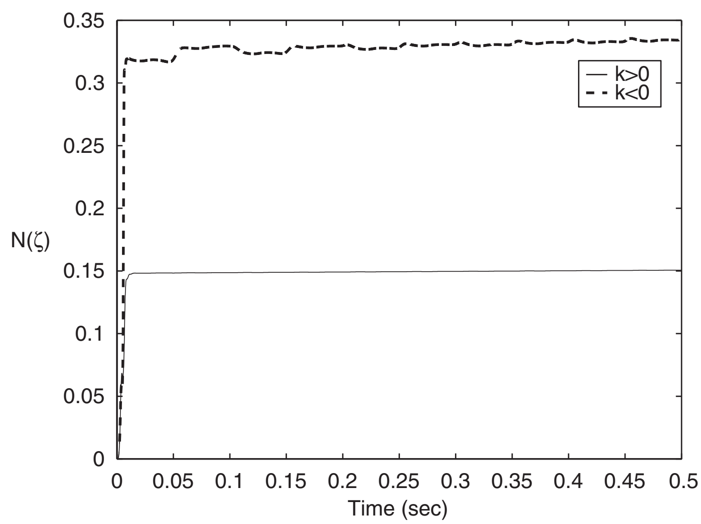

Figure 3. Nussbaum function.

$\Delta=\theta_0^*\psi(t)$ with θ_0^* being the unknown constant and $\psi(t)$ being the known function that is assumed here to be $\psi(t)=\sin(2\pi 10t)$. The high-frequency gain k is completely unknown, i.e. either positive or negative constant with unknown sign. In the simulation study, the unknown disturbance Δ is assumed to be mainly from the pivot nonlinearity, which was elegantly modeled in [20].

Let $y_d(t)$, $\dot{y}_d(t)$, $\ddot{y}_d(t)$ be the desired position, velocity and acceleration, respectively. Define the tracking error as $e(t)=y_d(t)-y(t)$ and the dynamic tracking error [21] as $r(t)=\dot{e}(t)+\lambda e(t)$

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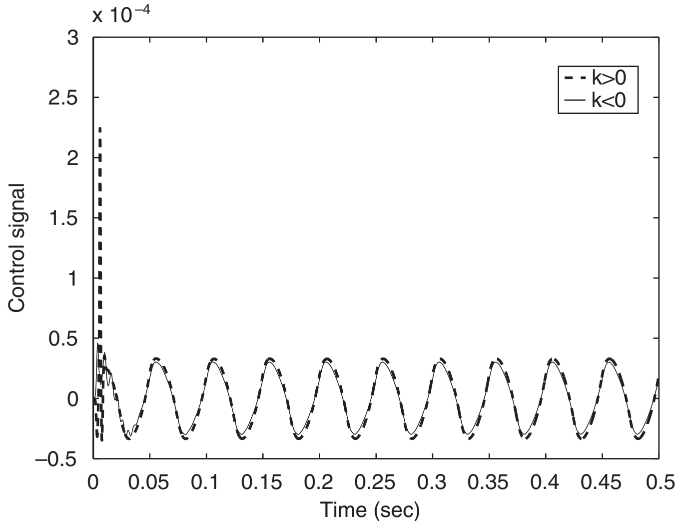

Figure 4. Control signal.

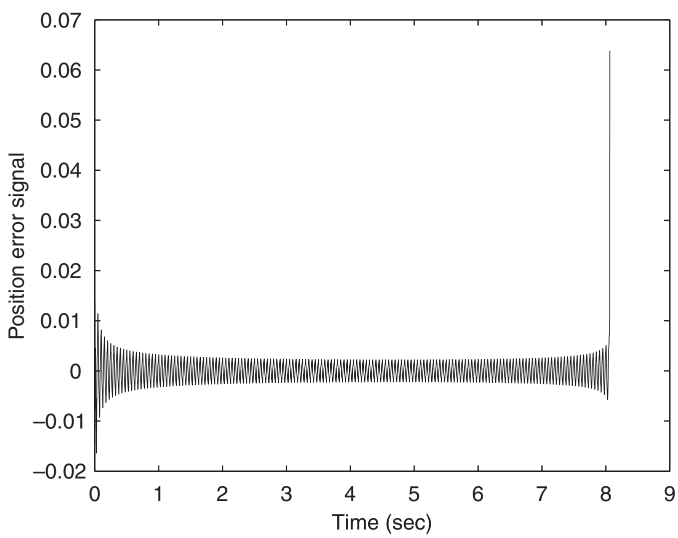

Figure 5. Position error signal without parameter adaptation.

with constant $\lambda > 0$. The reference velocity and acceleration are defined, respectively, as $\dot{y}_r(t) = \dot{y}_d(t) + \lambda e(t)$ and $\ddot{y}_r(t) = \ddot{y}_d(t) + \lambda \dot{e}(t)$.

Accordingly, the dynamic equation of system (53) can be represented in the following state-space form:

$$\dot{r} = gu + \Delta^* + \ddot{y}_r \tag{54}$$

where $g \triangleq -k$ and $\Delta^* \triangleq k\Delta = k\theta_0^* \psi = \theta^* \psi$.

Consider the following adaptive control:

$$\begin{aligned} u(t) &= N(\zeta)\left[k_I \int_0^t r(\tau)\,\mathrm{d}\tau + k_P r(t) + \hat{\theta}\psi + \ddot{y}_r\right] \\ \dot{\zeta} &= r(t)\left[k_I \int_0^t r(\tau)\,\mathrm{d}\tau + k_P r(t) + \hat{\theta}\psi + \ddot{y}_r\right] \\ \dot{\hat{\theta}} &= \gamma_\theta[r(t)\psi - \sigma_\theta\hat{\theta}] \end{aligned} \tag{55}$$

The R/W (Read/Write) head is driven by the VCM actuator to track a sinusoidal reference signal $y_d(t)=0.1\sin(2\pi 20t)$. The control parameters are chosen as $k_I=0.004$, $k_P=12\mathrm{e}-6$, $\lambda=800$, $\gamma_\theta=2$, $\sigma_\theta=0.01$.

The control algorithms are tested by changing the sign of the high-frequency gain k. The simulation results are given in Figures 1–4, which has shown that all the closed-loop signals are bounded for both cases (i.e. $k>0$ and $k<0$). To check the robustness of the scheme, the parameter adaptation has been removed and the system actually blows up at $t=8.06125$, as can be seen from Figure 5, which in turns verifies the proposed robust adaptive scheme.

5. CONCLUSION

In this paper, robust adaptive control has been presented for a class of perturbed uncertain strict-feedback nonlinear systems with unknown control coefficients. The design method does not require the *a priori* knowledge of the signs of the unknown control coefficients due to the incorporation of Nussbaum gain in the controller design. It has been proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals.

APPENDIX A

In order to prove Lemma 1, the following two lemmas are useful and introduced first.

Lemma A.1

If the continuous function $\zeta(t)$, $t\in\mathbb{R}^+$, is strictly monotonic in $[t_\alpha, t_\beta]\subset\mathbb{R}^+$, $t_\alpha<t_\beta$, then after applying integration by parts for a few times, the function

$$N_g(t,\zeta(t_\alpha),\zeta(t_\beta)) = \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} g_0 N(\zeta(\tau))\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\zeta(\tau) \tag{A1}$$

with $N(\zeta)=\zeta^2\cos(\zeta)$ can be calculated by

$$\begin{aligned} N_g(t,\zeta(t_\alpha),\zeta(t_\beta)) &= g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} + 2g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} \\ &\quad -2g_0\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} + \int_{t_\alpha}^{t_\beta} 2c_1 g_0\sin(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau \\ &\quad -\int_{t_\alpha}^{t_\beta} 2c_1 g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau - \int_{t_\alpha}^{t_\beta} c_1 g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau \end{aligned} \tag{A2}$$

Proof
Since $\zeta(t)$ is strictly monotonic in $[t_\alpha, t_\beta]$, there exists an inverse function $\zeta^{-1}(\cdot)$ that is also strictly monotonic and $\zeta^{-1}(\zeta(t)) \equiv t$. Noting $N(\zeta)=\zeta^2\cos(\zeta)$, (A1) can be re-written as

$$N_g(t,\zeta(t_\alpha),\zeta(t_\beta)) = \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} g_0\zeta^2\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta$$

Integrating by parts yields

$$\begin{aligned} N_g(t,\zeta(t_\alpha),\zeta(t_\beta)) &= \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} g_0\zeta^2\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}[\sin(\zeta)] \\ &= g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\Big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} - \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} g_0\sin(\zeta)\,\mathrm{d}\{\zeta^2\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\} \end{aligned} \quad \text{(A3)}$$

Noting that $\mathrm{d}\zeta^{-1}(\zeta)=\mathrm{d}\tau$ and $\mathrm{d}\{\zeta^2\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\} = 2\zeta\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta + c_1\zeta^2\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau$, Equation (A3) becomes

$$\begin{aligned} N_g(t,\zeta(t_\alpha),\zeta(t_\beta)) &= g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\Big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} - \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\zeta\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta \\ &\quad - \int_{t_\alpha}^{t_\beta} c_1 g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau \end{aligned} \quad \text{(A4)}$$

Integrating by parts for the term $\int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\zeta\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta$ in (A4), we have

$$\begin{aligned} \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\zeta\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta &= -2g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\Big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} \\ &\quad + \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\cos(\zeta)\mathrm{d}\{\zeta\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\} \end{aligned} \quad \text{(A5)}$$

Noting that $\mathrm{d}\{\zeta\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\} = \mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta + c_1\zeta\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\tau$, Equation (A5) becomes

$$\begin{aligned} \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\zeta\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta &= -2g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\Big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} \\ &\quad + \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta + \int_{t_\alpha}^{t_\beta} 2c_1 g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau \end{aligned} \quad \text{(A6)}$$

Substituting (A6) into (A4) yields

$$\begin{aligned} N_g(t,\zeta(t_\alpha),\zeta(t_\beta)) &= g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\Big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} + 2g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\Big|_{\zeta(t_\alpha)}^{\zeta(t_\beta)} \\ &\quad - \int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta - \int_{t_\alpha}^{t_\beta} 2c_1 g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau \\ &\quad - \int_{t_\alpha}^{t_\beta} c_1 g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau \end{aligned} \quad \text{(A7)}$$

Similarly, integrating by parts for the term $\int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta$ in (A7) by noting that $\mathrm{d}\{\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\}=c_1\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\tau$, we have

$$\int_{\zeta(t_\alpha)}^{\zeta(t_\beta)} 2g_0\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}\,\mathrm{d}\zeta$$

$$=2g_0\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}|_{\zeta(t_\alpha)}^{\zeta(t_\beta)}-\int_{t_\alpha}^{t_\beta} 2c_1g_0\sin(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau \tag{A8}$$

Substituting (A8) into (A7), we have

$$\begin{aligned}N_g(t,\zeta(t_\alpha),\zeta(t_\beta)) &= g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}|_{\zeta(t_\alpha)}^{\zeta(t_\beta)}+2g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}|_{\zeta(t_\alpha)}^{\zeta(t_\beta)}\\ &\quad -2g_0\sin(\zeta)\mathrm{e}^{-c_1[t-\zeta^{-1}(\zeta)]}|_{\zeta(t_\alpha)}^{\zeta(t_\beta)}+\int_{t_\alpha}^{t_\beta} 2c_1g_0\sin(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau\\ &\quad -\int_{t_\alpha}^{t_\beta} 2c_1g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau-\int_{t_\alpha}^{t_\beta} c_1g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau\end{aligned}$$

so that (A2) is proven. □

Lemma A.2
If the continuous function $\zeta(t)$, $t\in\mathbb{R}^+$, is strictly monotonic in $[t_\alpha,t_\beta]\subset\mathbb{R}^+$, $t_\alpha<t_\beta$, the following inequalities hold:

(i) $|\int_{t_\alpha}^{t_\beta} 2c_1g_0\sin(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau|\leqslant(t_\beta-t_\alpha)2c_1g_0$,
(ii) $|\int_{t_\alpha}^{t_\beta} 2c_1g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t-\tau)}\,\mathrm{d}\tau|\leqslant(t_\beta-t_\alpha)2c_1g_0\zeta_m$,
(iii) $|\mathrm{e}^{-c_1t}\int_{t_\alpha}^{t_\beta} c_1g_0\zeta^2\sin(\zeta)\mathrm{e}^{c_1\tau}\,\mathrm{d}\tau|\leqslant g_0\zeta_m^2[\mathrm{e}^{-c_1(t-t_\beta)}-\mathrm{e}^{-c_1(t-t_\alpha)}]$, where $\zeta_m=\max\{|\zeta(t_\alpha)|,|\zeta(t_\beta)|\}$.

Proof
Properties (i), (ii) and (iii) can be easily proved by applying integral inequality and the proof is omitted. □

Lemma 1 is re-produced here for clarity.

Lemma 1
Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions defined on $[0,t_f)$ with $V(t)\geqslant 0$, $\forall t\in[0,t_f)$, and $N(\zeta)=\zeta^2\cos(\zeta)$. If the following inequality holds:

$$0\leqslant V(t)\leqslant c_0+\mathrm{e}^{-c_1t}\int_0^t [g_0N(\zeta)+1]\dot{\zeta}\mathrm{e}^{c_1\tau}\,\mathrm{d}\tau \quad \forall t\in[0,t_f) \tag{A9}$$

where constant $c_1>0$, g_0 is a non-zero constant, and $c_0>0$ represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t g_0N(\zeta)\dot{\zeta}\,\mathrm{d}\tau$ are bounded on $[0,t_f)$.

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Proof

We first show that $\zeta(t)$ is bounded on $[0,t_f)$ by seeking a contradiction. Suppose that $\zeta(t)$ is unbounded and two cases should be considered: (i) $\zeta(t)$ has no upper bound and (ii) $\zeta(t)$ has no lower bound.

Case (*i*): $\zeta(t)$ has no upper bound on $[0,t_f)$. In this case, there must exist a monotonically increasing sequence $\{t_i\}$, $i=1,2,\ldots$, such that the sequence $\{\zeta(t_i)\}$ is monotonically increasing with $\zeta(t_1)>|\zeta(0)|\geqslant 0$, $\lim_{i\to+\infty} t_i=t_f$, and $\lim_{i\to+\infty}\zeta(t_i)=+\infty$.

According to the local Inverse Function Theorem [22, p. 211], if the continuously differentiable function $\zeta(\cdot)$ is strictly monotonic on an interval $[\alpha,\beta]$, i.e. $\forall a\in[\alpha,\beta]$, $\zeta'(a)\neq 0$, then there exists an inverse function $\zeta^{-1}(\cdot)$ that is also strictly monotonic and continuously differentiable.

Let us first assume a special case that $\zeta(t)$ is strictly monotonically increasing on $[0,t_i)$ such that there exists an inverse function $\zeta^{-1}(\cdot)$ that is also strictly monotonic increasing and $\zeta^{-1}(\zeta(\tau))\equiv\tau$. Invoking Lemma A.1 by noting that $N(\zeta)=\zeta^2\cos(\zeta)$, we have

$$\begin{aligned}
N_g(t_i,\zeta(0),\zeta(t_i)) &= \int_{\zeta(0)}^{\zeta(t_i)} g_0 N(\zeta(\tau))\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\zeta(\tau)\\
&= g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\big|_{\zeta(0)}^{\zeta(t_i)} + 2g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\big|_{\zeta(0)}^{\zeta(t_i)}\\
&\quad -2g_0\sin(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\big|_{\zeta(0)}^{\zeta(t_i)} + \int_0^{t_i} 2c_1 g_0\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau\\
&\quad -\int_0^{t_i} 2c_1 g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau - \int_0^{t_i} c_1 g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau
\end{aligned}$$

Noting that $\zeta^{-1}(\zeta(0))=0$ and $\zeta^{-1}(\zeta(t_i))=t_i$, $N_g(t_i,\zeta(0),\zeta(t_i))$ can be calculated by

$$\begin{aligned}
N_g(t_i,\zeta(0),\zeta(t_i)) &= g_0\zeta^2(t_i)\sin(\zeta(t_i)) - g_0\zeta^2(0)\sin(\zeta(0))\mathrm{e}^{-c_1 t_i}\\
&\quad +2g_0\zeta(t_i)\cos(\zeta(t_i)) - 2g_0\zeta(0)\cos(\zeta(0))\mathrm{e}^{-c_1 t_i}\\
&\quad -2g_0\sin(\zeta(t_i)) + 2g_0\sin(\zeta(0))\mathrm{e}^{-c_1 t_i} + \int_0^{t_i} 2c_1 g_0\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau\\
&\quad -\int_0^{t_i} 2c_1 g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau - \int_0^{t_i} c_1 g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau
\end{aligned} \tag{A10}$$

Using the inequalities (i), (ii) and (iii) in Lemma A.2 and noting that $\zeta(t_i)>|\zeta(0)|\geqslant 0$ since $\zeta(t)$ is strictly monotonically increasing, $\forall t\in[0,t_i)$, we have

$$\left|\int_0^{t_i} 2c_1 g_0\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau\right| \leqslant 2t_i c_1 g_0 \tag{A11}$$

$$\left|\int_0^{t_i} 2c_1 g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau\right| \leqslant 2t_i c_1 g_0\zeta(t_i) \tag{A12}$$

$$\left|\mathrm{e}^{-c_1 t_i}\int_0^{t_i} c_1 g_0\zeta^2\sin(\zeta)\mathrm{e}^{c_1\tau}\,\mathrm{d}\tau\right| \leqslant g_0\zeta^2(t_i)[1-\mathrm{e}^{-c_1 t_i}] \tag{A13}$$

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From (A10) and (A11)–(A13), the following inequalities hold:

$$N_g(t_i,\zeta(0),\zeta(t_i))\leqslant g_0\zeta^2(t_i)[\sin(\zeta(t_i))+1-\mathrm{e}^{-c_1t_i}]+f_1(\zeta(0),\zeta(t_i)) \tag{A14}$$

where

$$\begin{aligned}f_1(\zeta(0),\zeta(t_i))&=2g_0\zeta(t_i)\cos(\zeta(t_i))-2g_0\sin(\zeta(t_i))+2t_ic_1g_0\zeta(t_i)+2t_ic_1g_0\\&\quad -g_0\zeta^2(0)\sin(\zeta(0))\mathrm{e}^{-c_1t_i}-2g_0\zeta(0)\cos(\zeta(0))\mathrm{e}^{-c_1t_i}+2g_0\sin(\zeta(0))\mathrm{e}^{-c_1t_i}\end{aligned}$$

Noting (A9) and (A14), we have

$$\begin{aligned}0\leqslant V(t_i)&\leqslant c_0+\int_{\zeta(0)}^{\zeta(t_i)}g_0N(\zeta(\tau))\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\zeta(\tau)+\int_{\zeta(0)}^{\zeta(t_i)}\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\zeta(\tau)\\&\leqslant c_0+N_g(t_i,\zeta(0),\zeta(t_i))+(\zeta(t_i)-\zeta(0))\sup_{\tau\in[0,t_i]}\mathrm{e}^{-c_1(t_i-\tau)}\\&\leqslant c_0+g_0\zeta^2(t_i)[\sin(\zeta(t_i))+1-\mathrm{e}^{-c_1t_i}]+f_1(\zeta(0),\zeta(t_i))+\zeta(t_i)-\zeta(0)\\&=\zeta^2(t_i)\left\{g_0[\sin(\zeta(t_i))+1-\mathrm{e}^{-c_1t_i}]+\frac{1}{\zeta^2(t_i)}[c_0+f_1(\zeta(0),\zeta(t_i))+\zeta(t_i)-\zeta(0)]\right\}\end{aligned}$$

Taking the limit as $i\to+\infty$, hence $t_i\to t_f$, $\zeta(t_i)\to+\infty$, $f_1(\zeta(0),\zeta(t_i))/\zeta^2(t_i)\to 0$, we have

$$0\leqslant\lim_{i\to+\infty}V(t_i)\leqslant\lim_{i\to+\infty}g_0\zeta^2(t_i)[\sin(\zeta(t_i))+1-\mathrm{e}^{-c_1t_i}]$$

which, if $g_0>0$, draws a contradiction when $[\sin(\zeta(t_i))+1-\mathrm{e}^{-c_1t_i}]<0$, and, if $g_0<0$, draws a contradictions when $[\sin(\zeta(t_i))+1-\mathrm{e}^{-c_1t_i}]>0$. Therefore, $\zeta(t)$ is upper bounded on $[0,t_f)$.

Now, let us come back to the general case when $\zeta(t)$ is continuous but not strictly monotonically increasing on $[0,t_i)$. As such, assume that $\zeta(t)$ has n critical points $\{t_{c1},t_{c2},\ldots,t_{cn}\}\subset[0,t_i)$, i.e. $\mathrm{d}/\mathrm{d}t\zeta(t_{cj})=0$, $j=1,\ldots,n$. Although $\zeta(t)$ is not strictly monotonic on $[0,t_i)$, it is monotonic on the intervals $[0,t_{c1}]$, $[t_{c1},t_{c2}],\ldots,[t_{c,n-1},t_{cn}]$, and $[t_{cn},t_i]$, respectively, such that the inverse function $\zeta^{-1}(\cdot)$ exists on these intervals. Invoking Lemma A.1 by noting that $N(\zeta)=\zeta^2\cos(\zeta)$, we have

$$\begin{aligned}N_g(t_i,\zeta(0),\zeta(t_i))&=\int_{\zeta(0)}^{\zeta(t_i)}g_0N(\zeta(\tau))\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\zeta(\tau)\\&=\left(\int_{\zeta(0)}^{\zeta(t_{c1})}+\int_{\zeta(t_{c1})}^{\zeta(t_{c2})}+\cdots+\int_{\zeta(t_{c,n-1})}^{\zeta(t_{cn})}+\int_{\zeta(t_{cn})}^{\zeta(t_i)}\right)g_0N(\zeta(\tau))\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\zeta(\tau)\\&=g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\left(\Big|_{\zeta(0)}^{\zeta(t_{c1})}+\Big|_{\zeta(t_{c1})}^{\zeta(t_{c2})}+\cdots+\Big|_{\zeta(t_{c,n-1})}^{\zeta(t_{cn})}+\Big|_{\zeta(t_{cn})}^{\zeta(t_i)}\right)\\&\quad+2g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\left(\Big|_{\zeta(0)}^{\zeta(t_{c1})}+\Big|_{\zeta(t_{c1})}^{\zeta(t_{c2})}+\cdots+\Big|_{\zeta(t_{c,n-1})}^{\zeta(t_{cn})}+\Big|_{\zeta(t_{cn})}^{\zeta(t_i)}\right)\\&\quad-2g_0\sin(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\left(\Big|_{\zeta(0)}^{\zeta(t_{c1})}+\Big|_{\zeta(t_{c1})}^{\zeta(t_{c2})}+\cdots+\Big|_{\zeta(t_{c,n-1})}^{\zeta(t_{cn})}+\Big|_{\zeta(t_{cn})}^{\zeta(t_i)}\right)\\&\quad+\int_0^{t_i}2c_1g_0\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau-\int_0^{t_i}2c_1g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau\\&\quad-\int_0^{t_i}c_1g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau\end{aligned} \tag{A15}$$

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The first term on the right-hand side of (A15) can be calculated by

$$\begin{aligned}
& g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\big(\big|_{\zeta(0)}^{\zeta(t_{c1})}+\big|_{\zeta(t_{c1})}^{\zeta(t_{c2})}+\cdots+\big|_{\zeta(t_{c,n-1})}^{\zeta(t_{cn})}+\big|_{\zeta(t_{cn})}^{\zeta(t_i)}\big)\\
&\quad = g_0\zeta^2(t_{c1})\sin(\zeta(t_{c1}))\mathrm{e}^{-c_1(t_i-t_{c1})}-g_0\zeta^2(0)\sin(\zeta(0))\mathrm{e}^{-c_1 t_i}\\
&\qquad +g_0\zeta^2(t_{c2})\sin(\zeta(t_{c2}))\mathrm{e}^{-c_1(t_i-t_{c2})}-g_0\zeta^2(t_{c1})\sin(\zeta(t_{c1}))\mathrm{e}^{-c_1(t_i-t_{c1})}\\
&\qquad +\cdots+g_0\zeta^2(t_{cn})\sin(\zeta(t_{cn}))\mathrm{e}^{-c_1(t_i-t_{cn})}-g_0\zeta^2(t_{c,n-1})\sin(\zeta(t_{c,n-1}))\mathrm{e}^{-c_1(t_i-t_{c,n-1})}\\
&\qquad +g_0\zeta^2(t_i)\sin(\zeta(t_i))-g_0\zeta^2(t_{cn})\sin(\zeta(t_{cn}))\mathrm{e}^{-c_1(t_i-t_{cn})}\\
&\quad = g_0\zeta^2(t_i)\sin(\zeta(t_i))-g_0\zeta^2(0)\sin(\zeta(0))\mathrm{e}^{-c_1 t_i}
\end{aligned}$$

Similarly, the second and the third terms can be calculated by

$$\begin{aligned}
& 2g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\big(\big|_{\zeta(0)}^{\zeta(t_{c1})}+\big|_{\zeta(t_{c1})}^{\zeta(t_{c2})}+\cdots+\big|_{\zeta(t_{c,n-1})}^{\zeta(t_{cn})}+\big|_{\zeta(t_{cn})}^{\zeta(t_i)}\big)\\
&\quad = 2g_0\zeta(t_i)\cos(\zeta(t_i))-2g_0\zeta(0)\cos(\zeta(0))\mathrm{e}^{-c_1 t_i}\\
& -2g_0\sin(\zeta)\mathrm{e}^{-c_1[t_i-\zeta^{-1}(\zeta)]}\big(\big|_{\zeta(0)}^{\zeta(t_{c1})}+\big|_{\zeta(t_{c1})}^{\zeta(t_{c2})}+\cdots+\big|_{\zeta(t_{c,n-1})}^{\zeta(t_{cn})}+\big|_{\zeta(t_{cn})}^{\zeta(t_i)}\big)\\
&\quad = -2g_0\sin(\zeta(t_i))+2g_0\sin(\zeta(0))\mathrm{e}^{-c_1 t_i}
\end{aligned}$$

Now we are ready to have

$$\begin{aligned}
N_g(t_i,\zeta(0),\zeta(t_i)) = {} & g_0\zeta^2(t_i)\sin(\zeta(t_i))-g_0\zeta^2(0)\sin(\zeta(0))\mathrm{e}^{-c_1 t_i}\\
& +2g_0\zeta(t_i)\cos(\zeta(t_i))-2g_0\zeta(0)\cos(\zeta(0))\mathrm{e}^{-c_1 t_i}\\
& -2g_0\sin(\zeta(t_i))+2g_0\sin(\zeta(0))\mathrm{e}^{-c_1 t_i}+\int_0^{t_i}2c_1 g_0\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau\\
& -\int_0^{t_i}2c_1 g_0\zeta\cos(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau-\int_0^{t_i}c_1 g_0\zeta^2\sin(\zeta)\mathrm{e}^{-c_1(t_i-\tau)}\,\mathrm{d}\tau
\end{aligned}$$

which is the same as (A10).

Therefore, the rest of the analysis is the same as the case when $\zeta(t)$ is strictly monotonically increasing and the same conclusion can be drawn, i.e. $\zeta(t)$ is upper bounded on $[0,t_f]$.

Case (*ii*): $\zeta(t)$ has no lower bound on $[0,t_f)$. There must exist a monotonically increasing sequence $\{\underline{t}_i\}$, $i=1,2,\ldots$, such that the sequence $\{-\zeta(\underline{t}_i)\}$ is monotonically increasing with $-\zeta(\underline{t}_1)>|\zeta(0)|\geqslant 0$, $\lim_{i\to+\infty}\underline{t}_i=t_f$, and $\lim_{i\to+\infty}[-\zeta(\underline{t}_i)]=+\infty$.

Letting $\chi(t)=-\zeta(t)$, from (A9)

$$0\leqslant V(\underline{t}_i)\leqslant c_0-\int_{\chi(0)}^{\chi(\underline{t}_i)}g_0 N(-\chi(\tau))\mathrm{e}^{-c_1(\underline{t}_i-\tau)}\,\mathrm{d}\chi(\tau)-\int_{\chi(0)}^{\chi(\underline{t}_i)}\mathrm{e}^{-c_1(\underline{t}_i-\tau)}\,\mathrm{d}\chi(\tau) \tag{A16}$$

Noting that $N(\cdot)$ is an even function, i.e. $N(\chi)=N(-\chi)$, (A16) becomes

$$0\leqslant V(\underline{t}_i)\leqslant c_0-\int_{\chi(0)}^{\chi(\underline{t}_i)}g_0 N(\chi(\tau))\mathrm{e}^{-c_1(\underline{t}_i-\tau)}\,\mathrm{d}\chi(\tau)-\int_{\chi(0)}^{\chi(\underline{t}_i)}\mathrm{e}^{-c_1(\underline{t}_i-\tau)}\,\mathrm{d}\chi(\tau)$$

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where

$$-\int_{\chi(0)}^{\chi(\underline{t}_i)} g_0 N(\chi(\tau))e^{-c_1(\underline{t}_i-\tau)}\,d\chi(\tau) = -N_g(\underline{t}_i,\chi(0),\chi(\underline{t}_i))$$

$$\leqslant g_0\chi^2(\underline{t}_i)[-\sin(\chi(\underline{t}_i))+1-e^{-c_1\underline{t}_i}] + f_2(\chi(0),\chi(\underline{t}_i)) \qquad \text{(A17)}$$

with

$$f_2(\chi(0),\chi(\underline{t}_i)) = -2g_0\chi(\underline{t}_i)\cos(\chi(\underline{t}_i))+2g_0\sin(\chi(\underline{t}_i))+2\underline{t}_i c_1 g_0\chi(\underline{t}_i)+2\underline{t}_i c_1 g_0 + g_0\chi^2(0)\sin(\chi(0))e^{-c_1\underline{t}_i}+2g_0\chi(0)\cos(\chi(0))e^{-c_1\underline{t}_i}-2g_0\sin(\chi(0))e^{-c_1\underline{t}_i}$$

Therefore, it follows

$$0\leqslant V(\underline{t}_i)\leqslant c_0 - N_g(\underline{t}_i,\chi(0),\chi(\underline{t}_i)) - [\chi(\underline{t}_i)-\chi(0)]\inf_{\tau\in[0,\underline{t}_i]} e^{-c_1(\underline{t}_i-\tau)}$$

$$\leqslant c_0 + g_0\chi^2(\underline{t}_i)[-\sin(\chi(\underline{t}_i))+1-e^{-c_1\underline{t}_i}] + f_2(\chi(0),\chi(\underline{t}_i)) - [\chi(\underline{t}_i)-\chi(0)]e^{-c_1\underline{t}_i}$$

$$=\chi^2(\underline{t}_i)\left\{g_0[-\sin(\chi(\underline{t}_i))+1-e^{-c_1\underline{t}_i}] + \frac{1}{\chi^2(\underline{t}_i)}[c_0+f_2(\chi(0),\chi(\underline{t}_i))-(\chi(\underline{t}_i)-\chi(0))e^{-c_1\underline{t}_i}]\right\}$$

Taking the limit as $i\to+\infty$, hence $\underline{t}_i\to t_f$, $\chi(\underline{t}_i)\to+\infty$, $f_2(\chi(0),\chi(\underline{t}_i))/\chi^2(\underline{t}_i)\to 0$, we have

$$0\leqslant \lim_{i\to+\infty} V(\underline{t}_i)\leqslant \lim_{i\to+\infty}\chi^2(\underline{t}_i)g_0[-\sin(\chi(\underline{t}_i))+1-e^{-c_1\underline{t}_i}] \qquad \text{(A18)}$$

which, if $g_0>0$, draws a contradiction when $[-\sin(\chi(\underline{t}_i))+1-e^{-c_1\underline{t}_i}]<0$, and, if $g_0<0$, draws a contradictions when $[-\sin(\chi(\underline{t}_i))+1-e^{-c_1\underline{t}_i}]>0$. Therefore, $\zeta(t)$ is lower bounded on $[0,t_f)$.

Therefore, $\zeta(t)$ must be bounded on $[0,t_f)$. In addition, $V(t)$ and $\int_0^t g_0N(\zeta)\dot{\zeta}\,d\tau$ are bounded on $[0,t_f)$. □

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