Adaptive Neural Control for a Class of Uncertain Nonlinear Systems in Pure-Feedback Form With Hysteresis Input

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***Abstract*—In this paper, adaptive neural control is investigated for a class of unknown nonlinear systems in pure-feedback form with the generalized Prandtl–Ishlinskii hysteresis input. To deal with the nonaffine problem in face of the nonsmooth characteristics of hysteresis, the mean-value theorem is applied successively, first to the functions in the pure-feedback plant, and then to the hysteresis input function. Unknown uncertainties are compensated for using the function approximation capability of neural networks. The unknown virtual control directions are dealt with by Nussbaum functions. By utilizing Lyapunov synthesis, the closed-loop control system is proved to be semiglobally uniformly ultimately bounded, and the tracking error converges to a small neighborhood of zero. Simulation results are provided to illustrate the performance of the proposed approach.**

***Index Terms*—Adaptive control, hysteresis, neural networks (NNs), nonlinear systems, pure-feedback.**

I. Introduction

CONTROL of nonlinear systems with unknown hysteresis nonlinearities has been an active topic, since hysteresis nonlinearities are common in smart material-based actuators, such as piezoceramics and shape memory alloys. It is challenging to control a system with hysteresis nonlinearities, because they severely limit system performance such as giving rise to undesirable inaccuracy or oscillations and may even lead to instability [1]. In addition, due to the nonsmooth characteristics of hysteresis nonlinearities, traditional control methods are insufficient in dealing with the effects of unknown hysteresis. Therefore, advanced control techniques are much needed to mitigate the effects of hysteresis.

One of the most common approaches is to construct an inverse operator to cancel the effects of the hysteresis as in [1] and [2]. However, it is a challenging task to construct the inverse operator for the hysteresis due to the complexity and uncertainty of hysteresis. To circumvent these difficulties, alternative control approaches that do not need an inverse model have also been developed in [3]–[6]. In [3] and [4], robust adaptive control and adaptive backstepping control were investigated for a class of nonlinear system with unknown backlashlike hysteresis, respectively. In [5] and [6], adaptive variable structure control and adaptive backstepping control were proposed for a class of continuous-time nonlinear dynamic systems preceded by a hysteresis nonlinearity with the conventional Prandtl–Ishlinskii (P–I) model representation, respectively.

In this paper, we consider a class of unknown nonlinear systems in pure-feedback form preceded by a generalized P–I hysteresis input. Compared with the backlashlike hysteresis and the conventional P–I hysteresis model discussed in [3]–[6], the generalized P–I hysteresis model proposed in [7] can capture the hysteresis phenomenon more accurately and accommodate more general classes of hysteresis shapes by adjusting not only the density function but also the input function. However, the difficulty in dealing with the generalized P–I hysteresis model lies in the fact that the input function in the generalized P–I hysteresis model is unknown and nonaffine. Motivated by the works in [8]–[10], in this paper, we adopt the mean-value theorem to transform the unknown nonaffine input function to a partially affine form, which can be handled by extending some available techniques for affine nonlinear system control in the literature.

For pure-feedback systems, the cascade and nonaffine properties make it difficult to find the explicit virtual controls and the actual control to stabilize the pure-feedback systems. In [11] and [12], much simpler pure-feedback systems, where the last one or two equations were assumed to be affine, were discussed. In [13], an "ISS-modular" approach combined with the small-gain theorem was presented for adaptive neural control of the completely nonaffine pure-feedback system. In this paper, we also consider a class of unknown nonlinear systems in pure-feedback form. The nonaffine problem in the control variable and virtual ones is dealt with by adopting the mean-value theorem, motivated by the works in [8]–[10], without the assumptions that the last one or two equations are affine as in [11] and [12]. The unknown virtual control directions are dealt with by using Nussbaum functions.

Our main contributions in this paper are highlighted as follows.

1) To the best of our knowledge, it is the first time, in the literature, that the tracking control problem of unknown nonlinear systems in pure-feedback form with the generalized P–I hysteresis input is investigated.

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2) The difficulty in dealing with the generalized P–I hysteresis model, i.e., the nonaffine problem of the uncertain nonlinear input function in the generalized P–I hysteresis model, is solved by adopting the mean-value theorem.
3) Different from the previous works in [5] and [6], the σ-modification is included in the adaptation law of estimate of density function $\hat{p}(t, r)$ to establish the different closed-loop stability.
4) The combination of the mean-value theorem and Nussbaum functions is used to solve the nonaffine and unknown virtual control direction problems in the pure-feedback nonlinear systems, without the assumptions that the last one or two equations are affine as in [11] and [12].

The organization of this paper is as follows. The problem formulation and preliminaries are given in Section II. In Section III, adaptive neural control is developed for a class of unknown nonlinear systems in pure-feedback form with the uncertain generalized P–I hysteresis input. The closed-loop system stability is analyzed as well. Results of extensive simulation studies are shown to demonstrate the effectiveness of the approach in Section IV, followed by the conclusion in Section V.

II. Problem Formulation and Preliminaries

Throughout this paper, $(\tilde{\cdot}) = (\hat{\cdot}) - (\cdot)$, $\|\cdot\|$ denotes the two-norm, and $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and largest eigenvalues of a square matrix $(\cdot)$, respectively.

Definition 1: The solution $X(t)$ of (7) is semiglobally uniformly ultimately bounded (SGUUB) if, for any compact set Ω_0 and all $X(t_0) \in \Omega_0$, there exists an $\mu > 0$ and $T(\mu, X(t_0))$ such that $\|X(t)\| \leq \mu$ for all $t \geq t_0 + T$ [14].

A. Problem Formulation

Consider the following class of unknown nonlinear system in pure-feedback form whose input is preceded by the uncertain generalized P–I hysteresis:

$$\begin{aligned} \dot{x}_j &= f_j(\bar{x}_j, x_{j+1}), \qquad 1 \leq j \leq n-1 \\ \dot{x}_n &= f_n(\bar{x}_n, u) + d(t) \\ y &= x_1 \end{aligned} \tag{1}$$

where $\bar{x}_j = [x_1, \ldots, x_j]^{\mathrm{T}} \in R^j$ is the vector of states of the first j differential equations, and $\bar{x}_n = [x_1, \ldots, x_n]^{\mathrm{T}} \in R^n$; $f_j(\cdot)$ and $f_n(\cdot)$ are unknown smooth functions; $d(t)$ is a bounded disturbance; $y \in R$ is the output of the system; and $u \in R$ is the input of the system and the output of the hysteresis nonlinearity, which is represented by the generalized P–I model in [7] as follows:

$$\begin{aligned} u(t) &= h(v)(t) - \int_0^D p(r) F_r[v](t)dr \\ F_r[v](0) &= h_r\left(v(0), 0\right) \\ F_r[v](t) &= h_r\left(v(t), F_r[v](t_i)\right), \\ &\qquad \text{for} \quad t_i < t \leq t_{i+1}, \qquad 0 \leq i \leq N-1 \\ h_r(v, w) &= \max\left(v - r, \min(v + r, w)\right) \end{aligned} \tag{2}$$

where v is the input to the hysteresis model; $0 = t_0 < t_1 < \cdots < t_N = t_E$ is a partition of $[0, t_E]$ such that the function v is monotone on each of the subintervals $(t_i, t_{i+1}]$; $p(r)$ is a given density function satisfying $p(r) \geq 0$ with $\int_0^\infty rp(r)dr < \infty$; D is a constant so that the density function $p(r)$ vanishes for large values of D; $F_r[v](t)$ is known as the play operator; and $h(v)$ is the hysteresis input function that satisfies the following assumptions [7].

Assumption 1: The function $h : R \rightarrow R$ is odd, nondecreasing, and locally Lipschitz continuous and satisfies $\lim_{v\rightarrow\infty} h(v) \rightarrow \infty$ and $(dh(v)/dv) > 0$ for almost every $v \in R$.

Assumption 2: The growth of the hysteresis function $h(v)$ is smooth, and there exist positive constants h_0 and h_1 such that $0 < h_0 \leq (dh(v)/dv) \leq h_1$.

The objective is to design adaptive neural control $v(t)$ for systems (1) and (2) such that all signals in the closed-loop system are bounded, while the tracking error between the output y and some reference trajectory y_d converges to a neighborhood of zero.

Remark 1: The conventional P–I hysteresis model studied in [5] and [6] is only a special case of the generalized P–I hysteresis model. If we select the input function $h(v)(t) = p_0 v$ with $p_0 = \int_0^D p(r)dr$ in (2), then the generalized P–I hysteresis model becomes a conventional P–I hysteresis model $u(t) = p_0 v - \int_0^D p(r)F_r[v](t)dr$. For the conventional P–I hysteresis model, the different hysteresis shapes are formulated by adjusting the density function only. However, for the generalized P–I hysteresis model, both the density function and the input function can be adjusted to describe a more general class of hysteresis characteristics.

Remark 2: Compared with the conventional P–I hysteresis model, the difficulty in dealing with the generalized P–I hysteresis model lies in the fact that the input function $h(v)$ is unknown, which needs some new treatments. In this paper, motivated by the works in [8]–[10], we adopt the mean-value theorem to transform the unknown nonaffine input function to a partially affine form, which can be seen as a multiplication of a control term with a function of control. As such, we can extend the available techniques for affine nonlinear system control in the literature to solve our problem.

Remark 3: Although it appears possible to rewrite (1) and (2) into the nonaffine form $\dot{x} = f(x, v)$, it still cannot be directly handled by the method proposed by Ge and Zhang [9], in which the mean-value theorem and the implicit-function theorem were adopted to handle the nonaffine problem. The reason is that if we want to apply the mean-value theorem and the implicit-function theorem to a function, one requirement is that the first-order derivative of the function is not equal to zero. However, due to the nonsmooth characteristics of hysteresis, the function $f(x, v)$ transformed from (1) and (2) is nondifferentiable and thus does not satisfy the conditions of applying the mean-value theorem and the implicit-function theorem. Therefore, we only apply the mean-value theorem to the smooth functions in (1), namely, $f_j(\cdot)$, $f_n(\cdot)$, and the hysteresis input function $h(v)$. For the nonsmooth function $F_r[v](t)$ in (2), we will develop a new treatment later.

Remark 4: There are many physical processes whose dynamics can be described by nonlinear differential equations like (1) and (2). Examples include some chemical reaction processes such as the continuously stirred tank reactor (CSTR) system given in [15] and [16]. Within the tank reactor, two chemicals are mixed and react to produce compound A at a concentration C_a. The objective is to manipulate the coolant flow rate q_c to control the concentration C_a at a desired value. The system is a pure-feedback system, which is nonaffine in the control input q_c. According to [17] and [18], the control valve that controls the coolant flow rate q_c exhibits considerable hysteresis. Since the generalized P–I hysteresis model can capture the hysteresis phenomenon more accurately and accommodate more general classes of hysteresis shapes by adjusting both the density function and the input function, we can adopt the generalized P–I hysteresis model to represent the hysteresis nonlinearity between the coolant flow rate q_c and the aperture of the control valve v. Therefore, we can regard the CSTR system as a physical example of pure-feedback systems with input hysteresis like (1) and (2).

To facilitate control design later in Section III, the following assumptions are needed.

Assumption 3: The desired trajectory y_d and its time derivatives up to the nth order $y_d^{(n)}$ are continuous and bounded.

Based on Assumption 3, we define the trajectory vector $\bar{x}_{d(j+1)} = [y_d \quad \dot{y}_d \quad \cdots \quad y_d^{(j)}]^{\mathrm{T}}$, where $j = 1, \ldots, n-1$, which is a vector from y_d to its jth time derivative, $y_d^{(j)}$, which will be used in the subsequent control design.

Assumption 4: There exists an unknown constant d^* such that $|d(t)| \leq d^*$.

Assumption 5: There exist a known constant $p_{\max}$ such that $p(r) \leq p_{\max}$ for all $r \in [0, D]$.

Remark 5: It is reasonable to set an upper bound for the density function $p(r)$, based on its properties that $p(r) \geq 0$ with $\int_0^\infty rp(r)dr < \infty$.

According to the mean-value theorem [19], we can express $f_j(\cdot,\cdot)$ in (1) as follows:

$$f_j(\bar{x}_j, x_{j+1}) = f_j\left(\bar{x}_j, x_{j+1}^0\right) + \left.\frac{\partial f_j(\bar{x}_j, x_{j+1})}{\partial x_{j+1}}\right|_{x_{j+1}=x_{j+1}^{\theta_j}} \times \left(x_{j+1} - x_{j+1}^0\right), \quad 1 \leq j \leq n-1$$
$$f_n(\bar{x}_n, u) = f_n(\bar{x}_n, u^0) + \left.\frac{\partial f_n(\bar{x}_n, u)}{\partial u}\right|_{u=u^{\theta_n}} (u - u^0) \tag{3}$$

where $x_{j+1}^{\theta_j} = \theta_j x_{j+1} + (1-\theta_j)x_{j+1}^0$, with $0 < \theta_j < 1$, $1 \leq j \leq n-1$, and $u^{\theta_n} = \theta_n u + (1-\theta_n)u^0$, with $0 < \theta_n < 1$. By choosing $x_{j+1}^0 = 0$ and $u^0 = 0$, (3) can be written as

$$f_j(\bar{x}_j, x_{j+1}) = f_j(\bar{x}_j, 0) + \left.\frac{\partial f_j(\bar{x}_j, x_{j+1})}{\partial x_{j+1}}\right|_{x_{j+1}=x_{j+1}^{\theta_j}} \times x_{j+1}, \qquad 1 \leq j \leq n-1$$
$$f_n(\bar{x}_n, u) = f_n(\bar{x}_n, 0) + \left.\frac{\partial f_n(\bar{x}_n, u)}{\partial u}\right|_{u=u^{\theta_n}} u. \tag{4}$$

For convenience of analysis, we define $g_j(\bar{x}_j, x_{j+1}^{\theta_j}) = (\partial f_j(\bar{x}_j, x_{j+1})/\partial x_{j+1})|_{x_{j+1}=x_{j+1}^{\theta_j}}$ and $g_n(\bar{x}_n, u^{\theta_n}) = (\partial f_n(\bar{x}_n, u)/\partial u)|_{u=u^{\theta_n}}$, which are also unknown nonlinear functions. Substituting (4) into (1), we have

$$\dot{x}_j = f_j(\bar{x}_j, 0) + g_j(\bar{x}_j, x_{j+1}^{\theta_j})x_{j+1}, \qquad 1 \leq j \leq n-1$$
$$\dot{x}_n = f_n(\bar{x}_n, 0) + g_n(\bar{x}_n, u^{\theta_n})u + d(t)$$
$$y = x_1. \tag{5}$$

In addition, according to the mean-value theorem [19], there also exists a constant $\theta_0 (0 < \theta_0 < 1)$ such that the unknown input function $h(v)$ in (2) satisfies the following property:

$$h(v) = h(v^*) + \left.\frac{\partial h(\cdot)}{\partial v}\right|_{v=v^{\theta_0}} (v - v^*)$$

where $v^{\theta_0} = \theta_0 v + (1-\theta_0)v^*$. According to Assumptions 1, 2 and the implicit-function theorem [20], we can find v^* such that $h(v^*) = 0$. Defining

$$g_0(v^{\theta_0}) = \left.\frac{\partial h(\cdot)}{\partial v}\right|_{v=v^{\theta_0}}$$

we have

$$h(v) = g_0(v^{\theta_0})(v - v^*).$$

Therefore, we can rewrite (2) as

$$u(t) = g_0(v^{\theta_0})v - g_0(v^{\theta_0})v^* - \int_0^D p(r)F_r[v](t)dr. \tag{6}$$

Substituting (6) into (5) leads to our unified system

$$\dot{x}_j = f_j(\bar{x}_j, 0) + g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right)x_{j+1}, \qquad 1 \leq j \leq n-1$$
$$\dot{x}_n = f_n(\bar{x}_n, 0) + g_n(\bar{x}_n, u^{\theta_n}) \times \left[g_0(v^{\theta_0})v - g_0(v^{\theta_0})v^* - \int_0^D p(r)F_r[v](t)dr\right] + d(t)$$
$$y = x_1. \tag{7}$$

Assumption 6: There exist constants $\underline{g}_j$ and $\bar{g}_j$ such that $0 < \underline{g}_j \leq |g_j(\cdot)| \leq \bar{g}_j < \infty$ for $j = 1, \ldots, n$.

Remark 6: Assumption 6 implies that smooth functions $g_j(\cdot)$ for $j = 1, \ldots, n$ are strictly either positive or negative, which is reasonable because $g_j(\cdot)$, being away from zero, is the controllable condition of system (7), which is made in most control schemes [21], [22]. Without loss of generality, we shall assume that $g_n(\bar{x}_n, u^{\theta_n}) > 0$, while no knowledge is required for the signs of $g_j(\cdot)$, where $j = 1, 2, \ldots, n-1$.

B. RBFNN Approximation

In control engineering, the radial basis function neural network (RBFNN) has been successfully used as a linearly parameterized function approximator to achieve various objectives, such as modeling, identification, and feedback linearization, by virtue of its universal approximation capabilities, learning and adaptation, and parallel distributed structures [14], [23]–[26]. In this paper, the following RBFNN is used to approximate the continuous function $h(Z) : R^q \to R$:

$$h_{\mathrm{nn}}(Z, W) = W^T S(Z) \tag{8}$$

where the input vector $Z \in \Omega \subset R^q$, weight vector $W = [w_1, w_2, \ldots, w_l]^T \in R^l$, with the neural network (NN) node number $l > 1$; and $S(Z) = [s_1(Z), \ldots, s_l(Z)]^T$, with $s_i(Z)$ being chosen as the commonly used Gaussian functions, which have the form

$$s_i(Z) = \exp\left[\frac{-(Z-\mu_i)^{\mathrm{T}}(Z-\mu_i)}{\eta_i^2}\right], \qquad i=1,2,\ldots,l \tag{9}$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^{\mathrm{T}}$ is the center of the receptive field and η is the width of the Gaussian function.

It has been proven that network (8) can approximate any continuous function over a compact set $\Omega_Z \subset R^q$ as

$$h(Z) = h_{\mathrm{nn}}(Z, W^*) + \varepsilon(Z) \qquad \forall Z \in \Omega_Z \tag{10}$$

where W^* is the ideal NN weights and $\varepsilon(Z)$ is the NN approximation error [23].

Assumption 7: There exist ideal constant weights W^* such that $|\varepsilon(Z)| \le \varepsilon^*$ with constant $\varepsilon^* > 0$ for all $Z \in \Omega_Z$. Moreover, W^* is bounded by $\|W^*\| \le w_m$ on the compact set Ω_Z.

The ideal weights W^* are "artificial" quantities that are required for analytical purposes. According to the discussion in [14], W^* is defined as follows:

$$W^* = \arg\min_{(W)}\left[\sup_{Z\in\Omega_Z} |h_{\mathrm{nn}}(Z, W) - h(Z)|\right]$$

which is unknown and needs to be estimated in control design. Let $\hat{W}$ be the estimate of W^*, and let $\tilde{W} = \hat{W} - W^*$ be the weight estimation error.

Remark 7: Although RBFNN is employed in our control design, it can be replaced by other linearly parameterized function approximators such as high-order NNs, fuzzy systems, polynomials, splines, and wavelet networks without difficulty. For a unified framework of different approximation structures in adaptive approximation-based control, interested readers can refer to [27].

The following lemma is useful for establishing the stability properties of the closed-loop system.

Lemma 1: Let $V(\cdot)$, $\zeta(\cdot)$ be the smooth functions defined on $[0, t_f)$ with $V(t) \ge 0$, $\forall t \in [0, t_f)$, and let $N(\cdot)$ be an even smooth Nussbaum-type function [28]. If the following inequality holds:

$$V(t) \le c_0 + e^{-c_1 t}\int_0^t [g(\cdot)N(\zeta) + 1]\,\dot{\zeta} e^{c_1\tau} d\tau \qquad \forall t \in [0, t_f)$$

where c_0 represents some suitable constant, c_1 is a positive constant, and $g(\cdot)$ is a time-varying parameter which takes values in the unknown closed intervals $I = [l^-, l^+]$, with $0 \notin I$, and then $V(t)$, $\zeta(t)$, and $\int_0^t g(\cdot)N(\zeta)\dot{\zeta} d\tau$ must be bounded on $[0, t_f)$.

III. Control Design and Stability Analysis

In this section, we will investigate adaptive neural control for system (7) using the backstepping method [21] combined with NN approximation. The backstepping design procedure contains n steps and involves the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \ldots, n$, where α_i is a virtual control which shall be developed for the corresponding i-subsystem based on an appropriate Lyapunov function V_i. The control law $v(t)$ is designed in the last step to stabilize the entire closed-loop system and deal with the hysteresis term. The closed-loop system can be proved to be SGUUB by Lyapunov stability analysis.

Step 1): Since $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, the derivative of z_1 is

$$\begin{aligned}\dot{z}_1 &= f_1(\bar{x}_1, 0) + g_1\left(\bar{x}_1, x_2^{\theta_1}\right) x_2 - \dot{y}_d\\ &= f_1(\bar{x}_1, 0) + g_1\left(\bar{x}_1, x_2^{\theta_1}\right)(z_2 + \alpha_1) - \dot{y}_d\\ &= g_1\left(\bar{x}_1, x_2^{\theta_1}\right)(z_2 + \alpha_1) + Q_1(Z_1)\end{aligned} \tag{11}$$

where $Q_1(Z_1) = f_1(\bar{x}_1, 0) - \dot{y}_d$, with $Z_1 = [\bar{x}_1, \dot{y}_d] \in \Omega_{Z_1} \subset R^2$. To compensate for the unknown function $Q_1(Z_1)$, we can use RBFNN in Section II-B, $\hat{W}_1^T S(Z_1)$, with $\hat{W}_1 \in R^{l\times 1}$, $S(Z_1) \in R^{l\times 1}$, and the NN node number $l > 1$, to approximate the function $Q_1(Z_1)$ on the compact set Ω_{Z_1} as follows:

$$Q_1(Z_1) = \hat{W}_1^T S(Z_1) - \tilde{W}_1^T S(Z_1) + \varepsilon_1(Z_1) \tag{12}$$

where the approximation error $\varepsilon_1(Z_1)$ satisfies $|\varepsilon_1(Z_1)| \le \varepsilon_1^*$ with positive constant ε_1^*.

Substituting (12) into (11), we obtain

$$\dot{z}_1 = g_1\left(\bar{x}_1, x_2^{\theta_1}\right)(z_2 + \alpha_1) + \hat{W}_1^T S(Z_1) - \tilde{W}_1^T S(Z_1) + \varepsilon_1(Z_1). \tag{13}$$

Choose the following virtual control law and adaptation laws:

$$\alpha_1 = N(\zeta_1)\left[k_1 z_1 + \hat{W}_1^T S(Z_1)\right] \tag{14}$$

$$\dot{\zeta}_1 = k_1 z_1^2 + z_1 \hat{W}_1^T S(Z_1) \tag{15}$$

$$\dot{\hat{W}}_1 = \Gamma_1\left[z_1 S(Z_1) - \sigma_1 \hat{W}_1\right] \tag{16}$$

where $\Gamma_1 = \Gamma_1^T \in R^{l\times l} > 0$, $k_1 > 0$, and $\sigma_1 > 0$ are design parameters.

Consider the following Lyapunov function candidate:

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2}\tilde{W}_1^T \Gamma_1^{-1} \tilde{W}_1. \tag{17}$$

The time derivative of (17), along with (13)–(16), is

$$\begin{aligned}\dot{V}_1 &= z_1 \dot{z}_1 + \tilde{W}_1^T \Gamma_1^{-1} \dot{\hat{W}}_1\\ &\le -k_1 z_1^2 + \left[g_1\left(\bar{x}_1, x_2^{\theta_1}\right) N_1(\zeta_1) + 1\right]\dot{\zeta}_1\\ &\quad + g_1\left(\bar{x}_1, x_2^{\theta_1}\right) z_1 z_2 - \sigma_1 \tilde{W}_1^T \hat{W}_1 + |z_1|\varepsilon_1^*.\end{aligned} \tag{18}$$

By using Young's inequality, we obtain the following inequalities:

$$-\sigma_1 \tilde{W}_1^T \hat{W}_1 \le -\frac{\sigma_1 \|\tilde{W}_1\|^2}{2} + \frac{\sigma_1 \|W_1^*\|^2}{2} \quad (19)$$

$$|z_1|\varepsilon_1^* \le \frac{z_1^2}{4c_{11}} + c_{11}\varepsilon_1^{*2} \quad (20)$$

$$g_1\left(\bar{x}_1, x_2^{\theta_1}\right) z_1 z_2 \le \frac{z_1^2}{4c_{12}} + c_{12} g_1^2\left(\bar{x}_1, x_2^{\theta_1}\right) z_2^2 \quad (21)$$

with constant parameters $c_{11} > 0$ and $c_{12} > 0$. Substituting (19)–(21) into (18) results in

$$\begin{aligned}\dot{V}_1 \le& -\left(k_1 - \frac{1}{4c_{11}} - \frac{1}{4c_{12}}\right) z_1^2 \\ &+ \left[g_1\left(\bar{x}_1, x_2^{\theta_1}\right) N_1(\zeta_1) + 1\right]\dot{\zeta}_1 - \frac{\sigma_1\|\tilde{W}_1\|^2}{2} \\ &+ c_{12} g_1^2\left(\bar{x}_1, x_2^{\theta_1}\right) z_2^2 + \frac{\sigma_1\|W_1^*\|^2}{2} + c_{11}\varepsilon_1^{*2} \\ \le& -\gamma_1 V_1 + \left[g_1\left(\bar{x}_1, x_2^{\theta_1}\right) N_1(\zeta_1) + 1\right]\dot{\zeta}_1 \\ &+ \rho_1 + c_{12} g_1^2\left(\bar{x}_1, x_2^{\theta_1}\right) z_2^2 \end{aligned} \quad (22)$$

where γ_1 and ρ_1 are positive constants, which are defined as

$$\gamma_1 = \min\left\{2\left(k_1 - \frac{1}{4c_{11}} - \frac{1}{4c_{12}}\right), \frac{\sigma_1}{\lambda_{\max}(\Gamma_1^{-1})}\right\}$$

$$\rho_1 = \frac{\sigma_1\|W_1^*\|^2}{2} + c_{11}\varepsilon_1^{*2}.$$

Multiplying both sides of (22) by $e^{\gamma_1 t}$ yields

$$\begin{aligned}\frac{d}{dt}(V_1 e^{\gamma_1 t}) \le \rho_1 e^{\gamma_1 t} + \left[g_1\left(\bar{x}_1, x_2^{\theta_1}\right) N_1(\zeta_1) + 1\right]\dot{\zeta}_1 e^{\gamma_1 t} \\ + c_{12} g_1^2\left(\bar{x}_1, x_2^{\theta_1}\right) z_2^2 e^{\gamma_1 t}. \end{aligned} \quad (23)$$

Integrating (23) over $[0, t]$, we have

$$\begin{aligned} V_1 \le& \frac{\rho_1}{\gamma_1} + \left[V_1(0) - \frac{\rho_1}{\gamma_1}\right] e^{-\gamma_1 t} \\ &+ e^{-\gamma_1 t}\int_0^t \left[g_1\left(\bar{x}_1, x_2^{\theta_1}\right) N_1(\zeta_1) + 1\right]\dot{\zeta}_1 e^{\gamma_1 \tau} d\tau \\ &+ e^{-\gamma_1 t}\int_0^t c_{12} g_1^2\left(\bar{x}_1, x_2^{\theta_1}\right) z_2^2 e^{\gamma_1 \tau} d\tau \quad (24) \\ \le& \frac{\rho_1}{\gamma_1} + V_1(0) \\ &+ e^{-\gamma_1 t}\int_0^t \left[g_1\left(\bar{x}_1, x_2^{\theta_1}\right) N_1(\zeta_1) + 1\right]\dot{\zeta}_1 e^{\gamma_1 \tau} d\tau \\ &+ e^{-\gamma_1 t}\int_0^t c_{12} g_1^2\left(\bar{x}_1, x_2^{\theta_1}\right) z_2^2 e^{\gamma_1 \tau} d\tau. \quad (25) \end{aligned}$$

Noting Assumption 6, the last term of (25) $e^{-\gamma_1 t}\int_0^t c_{12} g_1^2(\bar{x}_1, x_2^{\theta_1}) z_2^2 e^{\gamma_1 \tau} d\tau$ has the following property:

$$\begin{aligned} e^{-\gamma_1 t}&\int_0^t c_{12} g_1^2\left(\bar{x}_1, x_2^{\theta_1}\right) z_2^2 e^{\gamma_1 \tau} d\tau \\ &\le e^{-\gamma_1 t}\int_0^t c_{12}\bar{g}_1^2 z_2^2 e^{\gamma_1 \tau} d\tau \\ &\le \bar{g}_1^2 \sup_{\tau\in[0,t]}\left[z_2^2(\tau)\right] e^{-\gamma_1 t}\int_0^t c_{12} e^{\gamma_1 \tau} d\tau \\ &\le \frac{c_{12}}{\gamma_1}\bar{g}_1^2 \sup_{\tau\in[0,t]}\left[z_2^2(\tau)\right] \end{aligned} \quad (26)$$

where $\bar{g}_1$ is the upper bound for $|g_1(\cdot)|$ as defined in Assumption 6. Therefore, if z_2 can be kept bounded over a finite time interval $[0, t_f)$, then we can obtain the boundedness of the term $e^{-\gamma_1 t}\int_0^t c_{12} g_1^2(\bar{x}_1, x_2^{\theta_1}) z_2^2 e^{\gamma_1 \tau} d\tau$. Furthermore, (25) can be written as

$$V_1 \le c_1 + e^{-\gamma_1 t}\int_0^t \left[g_1\left(\bar{x}_1, x_2^{\theta_1}\right) N_1(\zeta_1) + 1\right]\dot{\zeta}_1 e^{\gamma_1 \tau} d\tau \quad (27)$$

where $c_1 = (\rho_1/\gamma_1) + V_1(0) + (c_{12}/\gamma_1)\bar{g}_1^2 \sup_{\tau\in[0,t_f]}[z_2^2(\tau)]$. According to Lemma 1, we can conclude that V_1, ζ_1, $\hat{W}_1$, and $\int_0^t [g_1(\bar{x}_1, x_2^{\theta_1}) N_1(\zeta_1) + 1]\dot{\zeta}_1 e^{\gamma_1 \tau} d\tau$ are all bounded on $[0, t_f)$. According to Proposition 2 [29], $t_f = \infty$, we know that z_1 and $\hat{W}_1$ are SGUUB. The boundedness of z_2 will be dealt with in the following steps.

Step j ($2 \le j < n$)**:** The derivative of z_j is

$$\begin{aligned}\dot{z}_j &= \dot{x}_j - \dot{\alpha}_{j-1} \\ &= f_j(\bar{x}_j, 0) + g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) x_{j+1} - \dot{\alpha}_{j-1}. \end{aligned} \quad (28)$$

Since α_{j-1} is a function of $\bar{x}_{j-1}, \bar{x}_{dj}, \zeta_{j-1}, \hat{W}_1, \ldots, \hat{W}_{j-1}$, its derivative $\dot{\alpha}_{j-1}$ can be expressed as

$$\begin{aligned}\dot{\alpha}_{j-1} &= \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_k}\dot{x}_k + \phi_{j-1} \\ &= \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_k} f_k(\bar{x}_k, x_{k+1}) + \phi_{j-1} \end{aligned} \quad (29)$$

where

$$\phi_{j-1} = \frac{\partial\alpha_{j-1}}{\partial\zeta_{j-1}}\dot{\zeta}_{j-1} + \frac{\partial\alpha_{j-1}}{\partial\bar{x}_{dj}}\dot{\bar{x}}_{dj} + \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial\hat{W}_k}\dot{\hat{W}}_k \quad (30)$$

which is computable. As such, $\dot{\alpha}_{j-1}$ can be seen as a function of $\bar{x}_j, (\partial\alpha_{j-1}/\partial x_1), \ldots, (\partial\alpha_{j-1}/\partial x_{j-1}), \phi_{j-1}$. Furthermore, we can rewrite (28) as

$$\dot{z}_j = g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right)(z_{j+1} + \alpha_j) + Q_j(Z_j) \quad (31)$$

where $Z_j = [\bar{x}_j, (\partial\alpha_{j-1}/\partial x_1), \ldots, (\partial\alpha_{j-1}/\partial x_{j-1}), \phi_{j-1}] \in \Omega_{Z_j} \subset R^{2j}$, and $Q_j(Z_j) = f_j(\bar{x}_j, 0) - \dot{\alpha}_{j-1}$ is an unknown function that can be approximated by the RBFNN in Section II-B, $\hat{W}_j^T S(Z_j)$, on the compact set Ω_{Z_j} as

$$Q_j(Z_j) = \hat{W}_j^T S(Z_j) - \tilde{W}_j^T S(Z_j) + \varepsilon_j(Z_j) \tag{32}$$

where the approximation error $\varepsilon_j(Z_j)$ satisfies $|\varepsilon_j(Z_j)| \le \varepsilon_j^*$ with positive constant ε_j^*. Substituting (32) into (28), we obtain

$$\begin{aligned}\dot{z}_j = g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right)(z_{j+1}+\alpha_j)& \\ + \hat{W}_j^T S(Z_j) - \tilde{W}_j^T S(Z_j) + \varepsilon_j(Z_j).&\end{aligned} \tag{33}$$

The following virtual control law and adaptation laws are considered:

$$\alpha_j = N(\zeta_j)\left[k_j z_j + \hat{W}_j^T S(Z_j)\right] \tag{34}$$

$$\dot{\zeta}_j = k_j z_j^2 + z_j \hat{W}_j^T S(Z_j) \tag{35}$$

$$\dot{\hat{W}}_j = \Gamma_j\left[z_j S(Z_j) - \sigma_j \hat{W}_j\right] \tag{36}$$

where $\Gamma_j = \Gamma_j^T > 0$, k_j, and σ_j are positive constants.

Define the following Lyapunov function candidate:

$$V_j = \frac{1}{2} z_j^2 + \frac{1}{2}\tilde{W}_j^T \Gamma_j^{-1} \tilde{W}_j. \tag{37}$$

Similar to the procedures outlined in Step 1), with the help of Young's inequality, the derivative of V_j in (37), along with (33)–(36), can be obtained as

$$\begin{aligned}\dot{V}_j \le& -\left(k_j - \frac{1}{4c_{j1}} - \frac{1}{4c_{j2}}\right) z_j^2 \\ &+ \left[g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) N_j(\zeta_j) + 1\right]\dot{\zeta}_j \\ &- \frac{\sigma_j \|\tilde{W}_j\|^2}{2} + c_{j2} g_j^2\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) z_{j+1}^2 \\ &+ \frac{\sigma_j \|W_j^*\|^2}{2} + c_{j1}\varepsilon_j^{*2} \\ \le& -\gamma_j V_j + \left[g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) N_j(\zeta_j) + 1\right]\dot{\zeta}_j \\ &+ \rho_j + c_{j2} g_j^2\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) z_{j+1}^2\end{aligned} \tag{38}$$

where γ_j and ρ_j are positive constants defined as

$$\gamma_j = \min\left\{2\left(k_j - \frac{1}{4c_{j1}} - \frac{1}{4c_{j2}}\right), \frac{\sigma_j}{\lambda_{\max}\left(\Gamma_j^{-1}\right)}\right\} \tag{39}$$

$$\rho_j = \frac{\sigma_j \|W_j^*\|^2}{2} + c_{j1}\varepsilon_j^{*2}. \tag{40}$$

with constant parameters $c_{j1} > 0$ and $c_{j2} > 0$. Multiplying both sides of (38) by $e^{\gamma_j t}$ and integrating over $[0, t]$, we have

$$\begin{aligned}V_j \le& \frac{\rho_j}{\gamma_j} + \left[V_j(0) - \frac{\rho_j}{\gamma_j}\right] e^{-\gamma_j t} \\ &+ e^{-\gamma_j t}\int_0^t \left[g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) N_j(\zeta_j) + 1\right]\dot{\zeta}_j e^{\gamma_j \tau} d\tau \\ &+ e^{-\gamma_j t}\int_0^t c_{j2} g_j^2\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) z_{j+1}^2 e^{\gamma_j \tau} d\tau\end{aligned} \tag{41}$$

$$\begin{aligned}\le& \frac{\rho_j}{\gamma_j} + V_j(0) \\ &+ e^{-\gamma_j t}\int_0^t \left[g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) N_j(\zeta_j) + 1\right]\dot{\zeta}_j e^{\gamma_j \tau} d\tau \\ &+ e^{-\gamma_j t}\int_0^t c_{j2} g_j^2\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) z_{j+1}^2 e^{\gamma_j \tau} d\tau.\end{aligned} \tag{42}$$

Similarly, as discussed in Step 1), if z_{j+1} can be kept bounded over a finite time interval $[0, t_f)$, we can readily guarantee the boundedness of the extra term $e^{-\gamma_j t}\int_0^t c_{j2} g_j^2(\bar{x}_j, x_{j+1}^{\theta_j}) z_{j+1}^2 e^{\gamma_j \tau} d\tau$ in (42) as follows:

$$e^{-\gamma_j t}\int_0^t c_{j2} g_j^2\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) z_{j+1}^2 e^{\gamma_j \tau} d\tau \le \frac{c_{j2}}{\gamma_j}\bar{g}_j^2 \sup_{\tau\in[0,t]}\left[z_{j+1}^2(\tau)\right]. \tag{43}$$

Therefore, (42) can be written as

$$V_j \le c_j + e^{-\gamma_j t}\int_0^t \left[g_j\left(\bar{x}_j, x_{j+1}^{\theta_j}\right) N_j(\zeta_j) + 1\right]\dot{\zeta}_j e^{\gamma_j \tau} d\tau \tag{44}$$

where $c_j = (\rho_j/\gamma_j) + V_j(0) + (c_{j2}/\gamma_j)\bar{g}_j^2 \sup_{\tau\in[0,t_f]}[z_{j+1}^2(\tau)]$. Then, applying Lemma 1, the boundedness of V_j, ζ_j, $\hat{W}_j$, and $\int_0^t [g_j(\bar{x}_j, x_{j+1}^{\theta_j}) N_j(\zeta_j) + 1]\dot{\zeta}_j e^{\gamma_j \tau} d\tau$ can be readily obtained. The boundedness of z_{j+1} will be dealt with in Step $(j+1)$.

Step n): This is the final step, in which we will design the control input $v(t)$. Since $z_n = x_n - \alpha_{n-1}$, its derivative is given by

$$\begin{aligned}\dot{z}_n =& f_n(\bar{x}_n, 0) + g_n(\bar{x}_n, u^{\theta_n}) \\ &\times \left[g_0(v^{\theta_0})v - g_0(v^{\theta_0})v^* - \int_0^D p(r) F_r[v](t) dr\right] \\ &+ d(t) - \dot{\alpha}_{n-1} \\ =& g_n(\bar{x}_n, u^{\theta_n}) \\ &\times \left[g_0(v^{\theta_0})v - g_0(v^{\theta_0})v^* - \int_0^D p(r) F_r[v](t) dr\right] \\ &+ Q_n(Z_n) + d(t) \\ =& g_n\left(\bar{x}_n, u^{\theta_n}\right) \\ &\times \left[g_0(v^{\theta_0})v - g_0(v^{\theta_0})v^* - \int_0^D p(r) F_r[v](t) dr\right] \\ &+ \hat{W}_n^T S(Z_n) - \tilde{W}_n^T S(Z_n) + \varepsilon_n(Z_n) + d(t)\end{aligned} \tag{45}$$

where $\hat{W}_n^T S(Z_n)$ is used to approximate the unknown function $Q_n(Z_n) = f_n(x, 0) - \dot{\alpha}_{n-1}$ on the compact set $\Omega_{Z_n} \subset R^n$, with $Z_n = [\bar{x}_n, (\partial\alpha_{n-1}/\partial x_1), \ldots, (\partial\alpha_{n-1}/\partial x_{n-1}), \phi_{n-1}] \in \Omega_{Z_n} \subset R^{2n}$, and the approximation error $\varepsilon_n(Z_n)$ satisfies $|\varepsilon_n(Z_n)| \leq \varepsilon_n^*$, with ε_n^* being a positive constant.

Choose the following Lyapunov function candidate:

$$V_n = \frac{1}{2}z_n^2 + \frac{1}{2}\tilde{W}_n^T\Gamma_n^{-1}\tilde{W}_n + \frac{1}{2\gamma_d}\tilde{d}^2 + \frac{\bar{g}_n}{2\gamma_p}\int_0^D \tilde{p}^2(t,r)dr \tag{46}$$

where $\tilde{d} = \hat{d} - d^*$; $\tilde{p}(t,r) = \hat{p}(t,r) - p_{\max}$; $\hat{d}$ and $\hat{p}(t,r)$ are the estimates of the disturbance bound d^* and the density function of $p(r)$, respectively; $\Gamma_n = \Gamma_n^T > 0$; and γ_d and γ_p are positive constants.

The derivative of V_n defined in (46), along with (45), is

$$\begin{aligned}\dot{V}_n =& z_n g_n\left(\bar{x}_n, u^{\theta_n}\right)\left[g_0(v^{\theta_0})v - \int_0^D p(r)F_r[v](t)dr\right]\\ &- z_n g_n(\bar{x}_n, u^{\theta_n})g_0(v^{\theta_0})v^* + z_n\hat{W}_n^T S(Z_n)\\ &- z_n\tilde{W}_n^T S(Z_n) + z_n\varepsilon_n(Z_n) + z_n d(t) + \tilde{W}_n^T\Gamma_n^{-1}\dot{\tilde{W}}_n\\ &+ \frac{1}{\gamma_d}\tilde{d}\dot{\tilde{d}} + \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr. \end{aligned}\tag{47}$$

From Assumptions 2 and 6, we know that $|g_n(x, u^{\theta_n})g_0 v^*| \leq C$, where C is a positive constant. Due to $|\varepsilon_n(Z_n)| \leq \varepsilon_n^*$ and Assumption 4, (47) becomes

$$\begin{aligned}\dot{V}_n \leq& z_n g_n\left(\bar{x}_n, u^{\theta_n}\right)\left[g_0(v^{\theta_0})v - \int_0^D p(r)F_r[v](t)dr\right]\\ &+ z_n\hat{W}_n^T S(Z_n) - z_n\tilde{W}_n^T S(Z_n) + |z_n|(C+\varepsilon_n^*) + |z_n|d^*\\ &+ \tilde{W}_n^T\Gamma_n^{-1}\dot{\tilde{W}}_n + \frac{1}{\gamma_d}\tilde{d}\dot{\tilde{d}} + \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr. \end{aligned}\tag{48}$$

The following control laws and adaptation laws are proposed:

$$v = N(\zeta_n)\left[k_n z_n + \hat{W}_n^T S(Z_n) + \hat{d}\tanh\left(\frac{z_n}{\omega}\right)\right] + v_h \tag{49}$$

$$v_h = -\,\mathrm{sgn}(z_n)\int_0^D \frac{\hat{p}(t,r)}{h_0}\left|F_r[v](t)\right| dr \tag{50}$$

$$\dot{\zeta}_n = k_n z_n^2 + z_n\hat{W}_n^T S(Z_n) + z_n\hat{d}\tanh\left(\frac{z_n}{\omega}\right) \tag{51}$$

$$\dot{\hat{W}}_n = \Gamma_n\left[z_n S(Z_n) - \sigma_n\hat{W}_n\right] \tag{52}$$

$$\dot{\hat{d}} = \gamma_d\left[z_n\tanh\left(\frac{z_n}{\omega}\right) - \sigma_d\hat{d}\right] \tag{53}$$

$$\frac{\partial}{\partial t}\hat{p}(t,r) = \begin{cases} -\gamma_p\sigma_p\hat{p}(t,r), & \hat{p}(t,r) \geq p_{\max}\\ \gamma_p\left[|z_n||F_r[v](t)|\right. & \\ \left.\quad -\sigma_p\hat{p}(t,r)\right], & 0 \leq \hat{p}(t,r) < p_{\max}\end{cases} \tag{54}$$

where k_n, σ_n, σ_d, σ_p and ω are positive constants.

Remark 8: The term v_h in (49) is used to cancel the effect caused by the nondifferentiable hysteresis term $\int_0^D p(r)F_r[v](t)dr$. Due to the integral form of $\int_0^D p(r)F_r[v](t)dr$, we cannot make assumptions on its boundedness and thus cannot design the traditional robust adaptive control. However, considering that the density function $p(r)$ is not a function of time, it can be treated as a "parameter" of the hysteresis model, and an adaptation law can be developed to obtain an estimate of it [5], [6].

Substituting (49)–(53) into (48), and using Young's inequality and the following property of the hyperbolic tangent function $\tanh(\cdot)$ [30], [31]:

$$0 \leq |z_n| - z_n\tanh\left(\frac{z_n}{\omega}\right) \leq 0.2785\omega$$

we obtain

$$\begin{aligned}\dot{V}_n \leq& -\left(k_n - \frac{1}{4c_{n1}}\right)z_n^2\\ &+ \left[g_n(x,u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n) + 1\right]\dot{\zeta}_n\\ &- \frac{\sigma_n\|\tilde{W}_n\|^2}{2} - \frac{\sigma_d\tilde{d}^2}{2} + \frac{\sigma_n\|W_j^*\|^2}{2} + \frac{\sigma_d d^{*2}}{2}\\ &+ 0.2785\omega d^* + c_{n1}\left(\varepsilon_n^* + C\right)^2 + g_n(x,u^{\theta_n})\\ &\times\left[-g_0(v^{\theta_0})|z_n|\int_0^D \frac{\hat{p}(t,r)}{h_0}\left|F_r[v](t)\right| dr\right.\\ &\left.\quad - z_n\int_0^D p(r)F_r[v](t)dr\right] + \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \end{aligned}\tag{55}$$

where c_{n1} is a positive constant. According to Assumptions 2 and 5, the last two terms of (55) can be written as

$$\begin{aligned} & g_n(x,u^{\theta_n})\left[-g_0(v^{\theta_0})|z_n|\int_0^D \frac{\hat{p}(t,r)}{h_0}\left|F_r[v](t)\right| dr\right.\\ &\qquad\left. -z_n\int_0^D p(r)F_r[v](t)dr\right]\\ &+ \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr\\ &\leq g_n(x,u^{\theta_n})\left[-|z_n|\int_0^D \hat{p}(t,r)\left|F_r[v](t)\right| dr\right.\\ &\qquad\left. + |z_n|\int_0^D p_{\max}\left|F_r[v](t)\right| dr\right]\\ &+ \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr\\ &\leq -g_n(x,u^{\theta_n})|z_n|\int_0^D \tilde{p}(t,r)\left|F_r[v](t)\right| dr\\ &+ \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr. \end{aligned}\tag{56}$$

According to (54), the adaptation law for the estimate of density function $\hat{p}(t,r)$ comprises two cases due to the different regions where $\hat{p}(t,r)$ belong to. Therefore, we also need to consider the following two cases for the analysis of (56).

Case 1) For $r \in D_{\max} = \{r : \hat{p}(t,r) \geq p_{\max}\} \subset [0, D]$, according to (54), we have

$$\tilde{p}(t,r) \geq 0 \tag{57}$$

$$\frac{\partial}{\partial t}\hat{p}(t,r) = -\gamma_p \sigma_p \hat{p}(t,r). \tag{58}$$

Substituting (57) and (58) into (56), we have

$$\begin{aligned}
&- g_n(x, u^{\theta_n})|z_n| \int\limits_{r\in D_{\max}} \tilde{p}(t,r)\,|F_r[v](t)|\,dr \\
&+ \frac{\bar{g}_n}{\gamma_p} \int\limits_{r\in D_{\max}} \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \\
&\leq -\sigma_p \bar{g}_n \int\limits_{r\in D_{\max}} \tilde{p}(t,r)\hat{p}(t,r)dr.
\end{aligned} \tag{59}$$

Case 2) For $r \in D^c_{\max}$, which is the complement set of $D_{\max}$ in $[0, D]$, i.e., $0 \leq \hat{p}(t,r) < p_{\max}$, from (54), we have

$$\tilde{p}(t,r) < 0 \tag{60}$$

$$\frac{\partial}{\partial t}\hat{p}(t,r) = \gamma_p \left[|z_n||F_r[v](t)| - \sigma_p \hat{p}(t,r)\right]. \tag{61}$$

Substituting (60) and (61) into (56), we have

$$\begin{aligned}
&- g_n(x, u^{\theta_n})|z_n| \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\,|F_r[v](t)|\,dr \\
&+ \frac{\bar{g}_n}{\gamma_p} \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \\
&\leq -g_n(x, u^{\theta_n})|z_n| \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\,|F_r[v](t)|\,dr \\
&\quad + \bar{g}_n|z_n| \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\,|F_r[v](t)|\,dr \\
&\quad - \sigma_p \bar{g}_n \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\hat{p}(t,r) \\
&\leq -\sigma_p \bar{g}_n \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\hat{p}(t,r)dr.
\end{aligned} \tag{62}$$

Combining Case 1) with Case 2), (56) can be written as

$$\begin{aligned}
&g_n(x, u^{\theta_n}) \left[-g_0(v^{\theta_0})|z_n| \int_0^D \frac{\hat{p}(t,r)}{h_0}\,|F_r[v](t)|\,dr \right.\\
&\qquad \left. - z_n \int_0^D p(r) F_r[v](t)dr \right] \\
&+ \frac{\bar{g}_n}{\gamma_p} \int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \\
&\leq -g_n(x, u^{\theta_n})|z_n| \int\limits_{r\in D_{\max}} \tilde{p}(t,r)\,|F_r[v](t)|\,dr \\
&\quad + \frac{\bar{g}_n}{\gamma_p} \int\limits_{r\in D_{\max}} \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \\
&\quad - g_n(x, u^{\theta_n})|z_n| \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\,|F_r[v](t)|\,dr \\
&\quad + \frac{\bar{g}_n}{\gamma_p} \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \\
&= -\sigma_p \bar{g}_n \int\limits_{r\in D_{\max}} \tilde{p}(t,r)\hat{p}(t,r)dr \\
&\quad - \sigma_p \bar{g}_n \int\limits_{r\in D^c_{\max}} \tilde{p}(t,r)\hat{p}(t,r)dr \\
&= -\sigma_p \bar{g}_n \int_0^D \tilde{p}(t,r)\hat{p}(t,r)dr.
\end{aligned} \tag{63}$$

By Young's inequality, we have

$$-\sigma_p \bar{g}_n \tilde{p}(t,r)\hat{p}(t,r) \leq -\frac{\sigma_p \bar{g}_n}{2}\tilde{p}^2(t,r) + \frac{\sigma_p \bar{g}_n}{2} p^2_{\max}. \tag{64}$$

Integrating both sides of (64) over $[0, D]$ results in

$$\begin{aligned}
&-\sigma_{p_1} \bar{g}_n \int_0^D \tilde{p}(t,r)\hat{p}(t,r)dr \\
&\qquad \leq -\frac{\sigma_p \bar{g}_n}{2} \int_0^D \tilde{p}^2(t,r)dr + \frac{\sigma_p \bar{g}_n D}{2} p^2_{\max}.
\end{aligned} \tag{65}$$

Therefore, according to (65), we can rewrite (63) further as

$$\begin{aligned}
&g_n(x, u^{\theta_n}) \left[-g_0(v^{\theta_0})|z_n| \int_0^D \frac{\hat{p}(t,r)}{h_0}\,|F_r[v](t)|\,dr \right.\\
&\qquad \left. - z_n \int_0^D p(r) F_r[v](t)dr \right] \\
&+ \frac{\bar{g}_n}{\gamma_p} \int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \\
&\leq -\frac{\sigma_p \bar{g}_n}{2} \int_0^D \tilde{p}^2(t,r)dr + \frac{\sigma_p \bar{g}_n D}{2} p^2_{\max}.
\end{aligned} \tag{66}$$

Substituting (66) into (55), we have

$$\begin{aligned}\dot{V}_n \le & -(k_n - \frac{1}{4c_{n1}})z_n^2 \\ & + \left[g_n(x,u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n)+1\right]\dot{\zeta}_n - \frac{\sigma_n\|\tilde{W}_n\|^2}{2} \\ & - \frac{\sigma_d\tilde{d}^2}{2} - \frac{\sigma_p\bar{g}_n}{2}\int_0^D \tilde{p}^2(t,r)dr + \frac{\sigma_n\|W_j^*\|^2}{2} \\ & + \frac{\sigma_d d^{*2}}{2} + 0.2785\omega d^* + c_{n1}\left(\varepsilon_n^* + C\right)^2 + \frac{\sigma_p\bar{g}_n D}{2}p_{\max}^2 \\ \le & -\gamma_n V_n + \left[g_n(x,u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n)+1\right]\dot{\zeta}_n + \rho_n\end{aligned} \tag{67}$$

where γ_n and ρ_n are positive constants defined as

$$\gamma_n = \min\left\{2\left(k_n - \frac{1}{4c_{n1}}\right), \frac{\sigma_n}{\lambda_{\max}\left(\Gamma_n^{-1}\right)}, \sigma_d\gamma_d, \sigma_p\gamma_p\right\} \tag{68}$$

$$\begin{aligned}\rho_n = & \frac{\sigma_n\|W_n^*\|^2}{2} + \frac{\sigma_d d^{*2}}{2} + 0.2785\omega d^* + c_{n1}\left(\varepsilon_n^* + C\right)^2 \\ & + \frac{\sigma_p\bar{g}_n D}{2}p_{\max}^2.\end{aligned} \tag{69}$$

Multiplying both sides of (67) and integrating over $[0,t]$, we have

$$\begin{aligned}V_n \le & \frac{\rho_n}{\gamma_n} + \left[V_n(0) - \frac{\rho_n}{\gamma_n}\right]e^{-\gamma_n t} \\ & + e^{-\gamma_n t}\int_0^t \left[g_n(x,u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n)+1\right]\dot{\zeta}_n e^{\gamma_n\tau}d\tau \\ \le & c_n + e^{-\gamma_n t}\int_0^t \left[g_n(x,u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n)+1\right]\dot{\zeta}_n e^{\gamma_n\tau}d\tau\end{aligned} \tag{70}$$

where $c_n = (\rho_n/\gamma_n) + V_n(0)$. According to Assumptions 1, 2, and 6, we can regard $g_n(x,u)g_0(v)$ in (70) as $g(\cdot)$, which is a time-varying parameter and takes values in the known closed intervals $I = [h_0\underline{g}_n, h_1\bar{g}_n]$, with $0 \notin I$. Using Lemma 1, we can conclude that $V_n(t)$, $\zeta_n(t)$, and, hence, $z_n(t)$, $\hat{W}_n$, and $\hat{d}$ are SGUUB. From the boundedness of $z_n(t)$, the boundedness of the extra term $e^{-\gamma_{n-1}t}\int_0^t c_{(n-1)2}g_{n-1}^2(\bar{x}_{n-1}, x_n^{\theta_{n-1}})z_n^2 e^{\gamma_{n-1}\tau}d\tau$ at Step $(n-1)$ is readily obtained. Applying Lemma 1 for $(n-1)$ times backward, it can be seen from the aforementioned iterative design procedure that V_j, z_j, $\hat{W}_j$, and, hence, x_j are SGUUB on $[0,t_f)$.

Remark 8: In order to use Lemma 1 to establish closed-loop stability, we need to express $\dot{V}_n$ in the form of $\dot{V}_n = -\gamma_n V_n + [g_n(x,u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n)+1]\dot{\zeta}_n + \rho_n$ as in (67). Thus, we need to adopt the σ-modification form in the adaptation law of $\hat{p}(t,r)$ as in (54), which can improve the robustness as well. This is different from the previous works [5], [6], where no σ-modification was included since only the property $\dot{V} \le 0$ was to be obtained.

The following theorem shows the stability and control performance of the closed-loop adaptive system.

Theorem 1: Consider the closed-loop system consisting of the plant (1), preceded by unknown hysteresis nonlinearities (2), and the control laws and adaptation laws (49)–(54). Under Assumptions 1–6, and given any initial conditions $z_i(0), \hat{W}_i(0), \hat{d}(0) (i = 1,2,\ldots,n)$ belonging to Ω_0, the overall closed-loop neural control system is SGUUB in the sense that all of the signals are bounded. Specifically, the states and weights in the closed-loop system will remain in the compact set Ω defined by

$$\Omega = \left\{z_j, \tilde{W}_j, \tilde{d}\,\middle|\,|z_j| \le \sqrt{2\mu_j},\quad \|\tilde{W}_j\| \le \sqrt{\frac{2\mu_j}{\lambda_{\min}(\Gamma_j^{-1})}},\quad |\tilde{d}| \le \sqrt{2\gamma_d\mu_n},\quad j = 1,2,\ldots,n\right\} \tag{71}$$

and eventually converge to the compact set Ω_s defined by

$$\Omega_s = \left\{z_j, \tilde{W}_j, \tilde{d}\,\middle|\,|z_j| \le \sqrt{2\mu_j^*},\quad \|\tilde{W}_j\| \le \sqrt{\frac{2\mu_j^*}{\lambda_{\min}\left(\Gamma_j^{-1}\right)}},\quad |\tilde{d}| \le \sqrt{2\gamma_d\mu_n^*},\quad j = 1,2,\ldots,n\right\} \tag{72}$$

where

$$\begin{aligned}\mu_j &= c_j + c_{j0}, \qquad j = 1,2,\ldots n, \\ c_n &= \frac{\rho_n}{\gamma_n} + V_n(0), \\ V_n(0) &= \frac{1}{2}z_n^2(0) + \frac{1}{2}\tilde{W}_n^T(0)\Gamma_n^{-1}\tilde{W}_n(0) + \frac{1}{2\gamma_d}\tilde{d}_n^2(0) \\ &\quad + \frac{\bar{g}_n}{2\gamma_p}\int_0^D \tilde{p}^2(0,r)dr, \\ c_j &= \frac{\rho_j}{\gamma_j} + V_j(0) + \frac{2c_{j2}}{\gamma_j}\bar{g}_j^2(c_{j+1} + c_{j+1,0}), \\ V_j(0) &= \frac{1}{2}z_j^2(0) + \frac{1}{2}\tilde{W}_j^T(0)\Gamma_j^{-1}\tilde{W}_j(0), \qquad j = 1,\ldots,n-1, \\ \mu_j^* &= c_j' + c_{j0}, \qquad j = 1,2,\ldots n, \\ c_n' &= \frac{\rho_n}{\gamma_n}, \\ c_j' &= \frac{\rho_j}{\gamma_j} + \frac{2c_{j2}}{\gamma_j}\bar{g}_j^2(c_{j+1} + c_{j+1,0}), \qquad j = 1,\ldots,n-1\end{aligned}$$

and with c_{j0} being the upper bound of $e^{-\gamma_j t}\int_0^t [g_j(\bar{x}_j, x_{j+1}^{\theta_j})N_j(\zeta_j)+1]\dot{\zeta}_j e^{\gamma_j\tau}d\tau$, where $j = 1,2,\ldots,n$.

Proof: For any given initial compact set Ω_0, i.e., $\{z_i(0), \hat{W}_i(0), \hat{d}(0)\} \in \Omega_0 (i = 1,2,\ldots,n)$, we can always construct a corresponding compact set Ω_{NN} comprising $\Omega_{Z_1},\ldots,\Omega_{Z_n}$, which is larger than Ω_0 and can be as large as we want, on which the NN approximation is valid. Based on the previous iterative derivation procedures from Step 1) to Step n) of backstepping, from (27), (44), and (70), and according to Lemma 1, we can conclude that V_j, z_j, $\hat{W}_j$, $\hat{d}$, and, hence, x_j are SGUUB, $i = 1, 2, \ldots, n$, i.e., all the signals in the closed-loop system are bounded.

Noting the definition of V_n in (46), and letting c_{n0} be the upper bound of the term $e^{-\gamma_n t}\int_0^t [g_n(x, u^{\theta_n}) g_0 N_n(\zeta_n) + 1] \dot{\zeta}_n e^{\gamma_n \tau} d\tau$, $c_n = (\rho_n/\gamma_n) + V_n(0)$, and $\mu_n = c_n + c_{n0}$ in (70), we have

$$|z_n| \leq \sqrt{2\mu_n} \qquad \|\tilde{W}_n\| \leq \sqrt{\frac{2\mu_n}{\lambda_{\min}\left(\Gamma_n^{-1}\right)}} \qquad |\tilde{d}| \leq \sqrt{2\gamma_d \mu_n}.$$

Similarly, in the rest of the steps from $(n-1)$ to 1), letting c_{j0} be the upper bound of $e^{-\gamma_j t}\int_0^t [g_j(\bar{x}_j, x_{j+1}^{\theta_j}) N_j(\zeta_j) + 1] \dot{\zeta}_j e^{\gamma_j \tau} d\tau$, $\quad c_j = (\rho_j/\gamma_j) + V_j(0) + (2c_{j2}/\gamma_j)\bar{g}_j^2(c_{j+1} + c_{j+1,0})$, and $\mu_j = c_j + c_{j0}$ in (44), we can obtain

$$|z_j| \leq \sqrt{2\mu_j} \quad \|\tilde{W}_i\| \leq \sqrt{\frac{2\mu_j}{\lambda_{\min}\left(\Gamma_j^{-1}\right)}}, \qquad j = 1, \ldots, n-1.$$

Furthermore, we can rewrite (70) as

$$V_n \leq \mu_n^* + \left[V_n(0) - \frac{\rho_n}{\gamma_n}\right] e^{-\gamma_n t}$$

where $\mu_n^* = c_n' + c_{n0}$, $c_n' = (\rho_n/\gamma_n)$, and c_{n0} is the upper bound of the term $e^{-\gamma_n t}\int_0^t [g_n(x, u^{\theta_n}) g_0 N_n(\zeta_n) + 1] \dot{\zeta}_n e^{\gamma_n \tau} d\tau$. As $t \to \infty$, we have

$$V_n \leq \mu_n^*.$$

Therefore, based on the definition of V_n in (46), we can conclude that when $t \to \infty$, the following inequalities are true:

$$|z_n| \leq \sqrt{2\mu_n^*} \quad \|\tilde{W}_n\| \leq \sqrt{\frac{2\mu_n^*}{\lambda_{\min}\left(\Gamma_n^{-1}\right)}} \quad |\tilde{d}| \leq \sqrt{2\gamma_d \mu_n^*}.$$

A similar conclusion can be made about z_j and $\hat{W}_j$ as follows:

$$|z_j| \leq \sqrt{2\mu_j^*} \quad \|\tilde{W}_j\| \leq \sqrt{\frac{2\mu_j^*}{\lambda_{\min}\left(\Gamma_j^{-1}\right)}}, \qquad j = 1, \ldots, n-1$$

with $\mu_j^* = c_j' + c_{j0}$ and $c_j' = (\rho_j/\gamma_j) + (2c_{j2}/\gamma_j)\bar{g}_j^2(c_{j+1} + c_{j+1,0})$ as $t \to \infty$.

In addition, from the definition of the bounds of the compact sets Ω in (71) and Ω_s in (72), and the definitions of γ_j and ρ_j in (39) and (40) and γ_n and ρ_n in (68) and (69), respectively, we can see that the size of the compact sets Ω and Ω_s depends on the choice of control parameters ω, $\lambda_{\max}(\Gamma_j^{-1})$, $\lambda_{\max}(\Gamma_n^{-1})$, k_j, k_n, γ_d and γ_p. In particular, by decreasing ω, $\lambda_{\max}(\Gamma_j^{-1})$, $\lambda_{\max}(\Gamma_n^{-1})$, and increasing k_j, k_n, γ_d, and γ_p, we can reduce μ_j, μ_j^*, μ_n, and μ_n^*, and thus, the size of the compact sets Ω and Ω_s will decrease. Therefore, as long as the initial conditions start in Ω_0, there exist some control parameters such that the states and weights will remain in the conservative compact set Ω and finally converge to the compact set Ω_s. Both of them belong to the chosen compact set Ω_{NN}. This completes the proof. ■

IV. Simulation Studies

In this section, simulation studies are presented to demonstrate the effectiveness of the proposed adaptive NN approach to deal with uncertain nonlinear systems in pure-feedback form preceded by the generalized P–I hysteresis.

Consider the following second-order nonlinear system with the generalized P–I hysteresis:

$$\begin{aligned} \dot{x}_1 &= x_2 + 0.05\sin(x_2) \\ \dot{x}_2 &= \frac{1 - e^{-x_2}}{1 + e^{-x_2}} + u + 0.1\sin(u) + 0.1\sin(6t) \\ y &= x_1 \end{aligned} \tag{73}$$

where u represents the output of the hysteresis described by the generalized P–I model $u(t) = h(v)(t) - \int_0^D p(r) F_r[v](t) dr$, with the density function $p(r) = 0.08e^{-0.0024(r-1)^2}$, $r \in [0, 100]$, and $h(v)(t) = 0.4(|v|\arctan(v) + v)$. We can check that plant (73) satisfies Assumptions 1–6. Our objective is to make the output of system (73), y, to track the desired trajectory, $y_d = 0.8\sin(0.5t) + 0.1\cos(t)$.

We adopt the control law and adaptation laws designed in Section III in the following:

$$\begin{aligned} \alpha_1 &= N(\zeta_1)\left[k_1 z_1 + \hat{W}_1^T S(Z_1)\right] \\ v &= N(\zeta_2)\left[k_2 z_2 + \hat{W}_2^T S(Z_2) + \hat{d}\tanh\left(\frac{z_2}{\omega}\right)\right] + v_h \\ v_h &= -\operatorname{sign}(z_2)\int_0^D \frac{\hat{p}(t,r)}{h_0}\left|F_r[v](t)\right| dr \\ \dot{\zeta}_1 &= k_1 z_1^2 + z_1 \hat{W}_1^T S(Z_1) \\ \dot{\zeta}_2 &= k_2 z_2^2 + z_2 \hat{W}_2^T S(Z_2) + z_2 \hat{d}\tanh\left(\frac{z_2}{\omega}\right) \\ \dot{\hat{W}}_1 &= \Gamma_1\left[z_1 S(Z_1) - \sigma_1 \hat{W}_1\right] \\ \dot{\hat{W}}_2 &= \Gamma_2\left[z_2 S(Z_2) - \sigma_2 \hat{W}_2\right] \\ \dot{\hat{d}} &= \gamma_d\left[z_2\tanh\left(\frac{z_2}{\omega}\right) - \sigma_d \hat{d}\right] \end{aligned}$$

$$\frac{\partial}{\partial t}\hat{p}(t,r) = \begin{cases} -\gamma_p\sigma_p\hat{p}(t,r), & \hat{p}(t,r) \geq p_{\max} \\ \gamma_p\left[|z_2||F_r[v](t)| - \sigma_p\hat{p}(t,r)\right], & 0 \leq \hat{p}(t,r) < p_{\max} \end{cases}$$

where $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$. The Nussbaum function is chosen as $N(\zeta) = \exp(\zeta^2)\cos((\pi/2)\zeta)$. The inputs

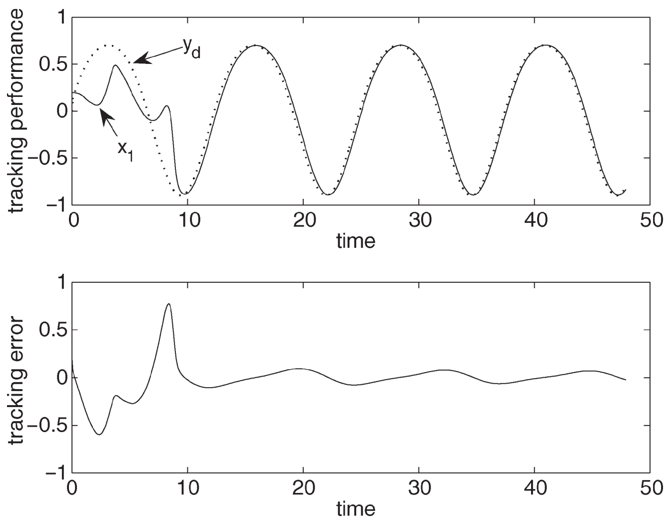

Fig. 1. Tracking performance.

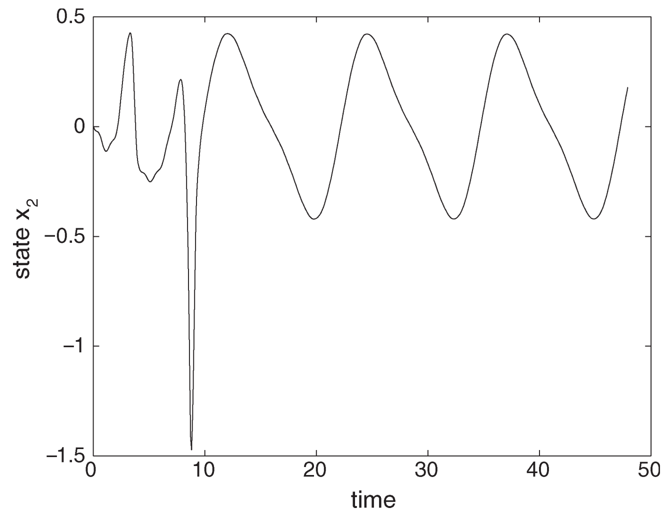

Fig. 2. State x_2.

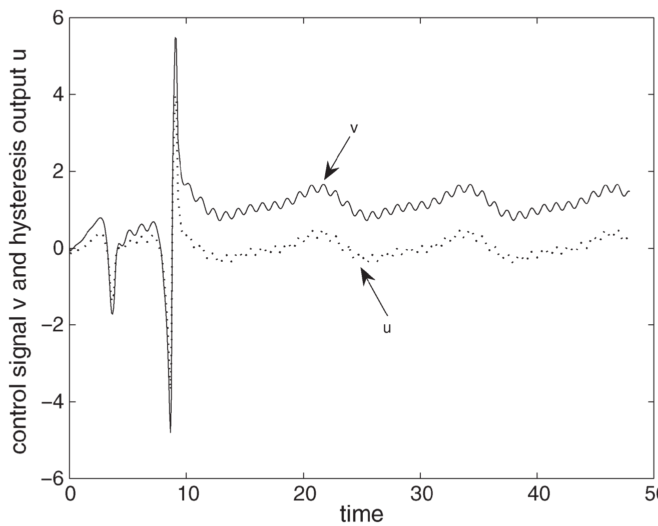

Fig. 3. Control signal and hysteresis output.

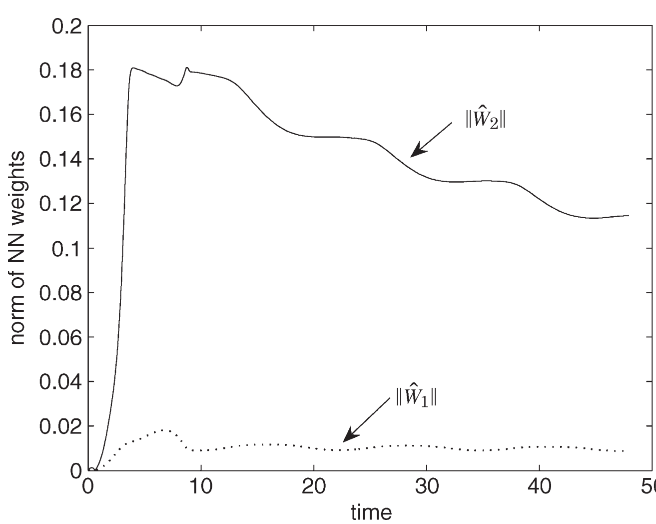

Fig. 4. Norm of NN weights.

of the NNs are $Z_1 = [x_1, y_d] \in R^2$ and $Z_2 = [x_1, x_2, (\partial\alpha_1/x_1), \phi_1] \in R^4$, where $\phi_1 = (\partial\alpha_1/\partial\zeta_1)\dot{\zeta}_1 + (\partial\alpha_1/\partial y_d)\dot{y}_d + (\partial\alpha_1/\partial\hat{W}_1)\dot{\hat{W}}_1$. The following initial conditions and control design parameters are chosen as $x_1(0) = 0.2$, $x_2(0) = \zeta_1(0) = \zeta_2(0) = \hat{d}(0) = 0.0$, $\hat{W}_1(0) = \hat{W}_2(0) = 0.0$, $k_1 = k_2 = 1.0$, $\Gamma_1 = 0.01I_{25}$, $\sigma_1 = 0.0$, $\Gamma_2 = 0.2I_{256}$, $\sigma_2 = 0.002$, $\sigma_p = 0.2$, $\gamma_p = 0.06$, $p_{\max} = 0.1$, $\omega = 0.1$, and $h_0 = 0.35$.

In practice, the selection of the centers and widths of RBF has a great influence on the performance of the designed controller. According to [23], Gaussian RBFNNs arranged on a regular lattice on R^n can uniformly approximate sufficiently smooth functions on closed bounded subsets. Accordingly, in the following simulation studies, the centers and widths are chosen on a regular lattice in the respective compact sets. Specifically, we employ five nodes for each input dimension of $\hat{W}_1^T S(Z_1)$ and four nodes for each input dimension of $\hat{W}_2^T S(Z_2)$; thus, we end up with 25 nodes (i.e., $l_1 = 25$) with centers $\mu_l = 1.0$ $(l = 1, 2, \ldots, l_1)$ evenly spaced in $[-4.0, +4.0] \times [-4.0, +4.0]$ and widths $\eta_l = 1.0$ $(l = 1, 2, \ldots l_1)$ for NN $\hat{W}_1^T S(Z_1)$, and 256 nodes (i.e., $l_2 = 256$) with centers μ_l $(l = 1, 2, \ldots, l_2)$ evenly spaced in $[-4.0, +4.0] \times [-4.0, +4.0] \times [-4.0, +4.0] \times [-4.0, +4.0]$ and widths $\eta_l = 1.0$ $(l = 1, 2, \ldots, l_2)$ for NN $\hat{W}_2^T S(Z_2)$.

Due to the use of sign function $\mathrm{sgn}(\cdot)$, the control signal v_h (50) becomes discontinuous, which may excite unmodeled high-frequency plant dynamics and cause the chattering phenomenon. To avoid the undesired chattering phenomenon, we will replace the sign function in v_h with the following saturation function in the simulation:

$$\mathrm{sat}(*) = \begin{cases} 1, & \text{if } * \geq \epsilon \\ \frac{*}{\epsilon}, & \text{if } |*| < \epsilon \\ 1, & \text{if } * > \epsilon \end{cases}$$

where ϵ is a small positive constant and chosen as 0.05 in this paper.

The simulation results are shown in Figs. 1–6. From Fig. 1, we observe that good tracking performance is achieved and that the tracking error converges to a small neighborhood of zero in less than one period of oscillation. At the same time, other signals, including the state x_2, control signal v, hysteresis output u, NN weight norms $\|\hat{W}_1\|$ and $\|\hat{W}_2\|$, Nussbaum function signals $\zeta_1, \zeta_2, N(\zeta_1)$, and $N(\zeta_2)$, and the disturbance parameter

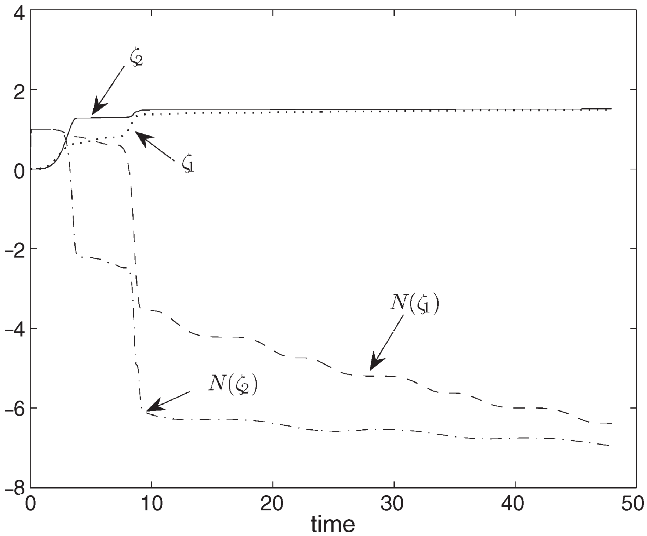

Fig. 5. Nussbaum function signals.

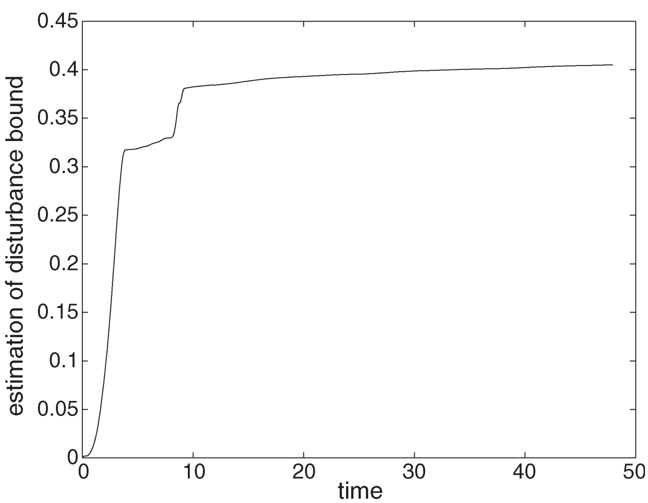

Fig. 6. Estimation of disturbance bound $\hat{d}$.

estimate $\hat{d}$ are kept bounded, as seen in Figs. 2–6. It is noted that there is a large difference between the signals v and u in Fig. 3, which indicates the significant hysteresis effect. In particular, in all figures, there are two obvious spikes at around 4 and 8 s, which result from the Nussbaum functions $N(\zeta_1)$ and $N(\zeta_2)$.

V. Conclusion

Adaptive neural control has been proposed for a class of unknown nonlinear systems in pure-feedback form preceded by the uncertain generalized P–I hysteresis. We adopted the mean-value theorem to solve the nonaffine problem in both system unknown nonlinear functions and unknown input function in the generalized P–I hysteresis model, and used Nussbaum function to deal with the problem of the unknown virtual control directions. The closed-loop control system has been theoretically shown to be SGUUB using the Lyapunov synthesis method. Simulation results have verified the effectiveness of the proposed approach.

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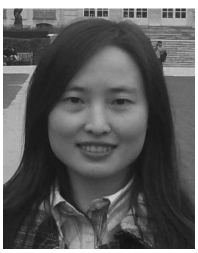

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