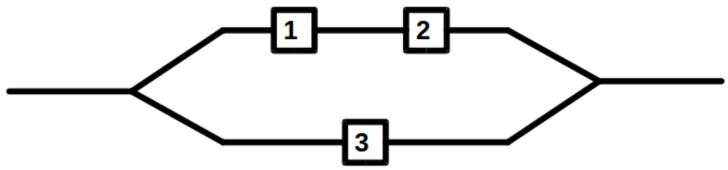
EX 2.4.1: Consider the following system of connected components shown below.

Components 1 & 2 are connected in series with component 3 connected to them in parallel.

For the entire system to function, either components 1 & 2 must function or component 3 must function.

The experiment consists of observing each component's condition and labeling it as either (S)uccess or (F)ailure.

So, for example, the outcome for components 1 & 2 functioning but 3 not functioning would be denoted as SSF.



(a) Determine the sample space Ω for the experiment.

$\Omega =$ (The set of all possible outcomes) $= \boxed{\{FFF,\ FFS,\ FSF,\ FSS,\ SFF,\ SSF,\ SFS,\ SSS\}}$

(b) Find the probability that Component 1 functions.

Let event $E_1 \equiv$ "Component 1 functions" $= \{SFF,\ SSF,\ SFS,\ SSS\}$

Then $\mathbb{P}(E_1) = \dfrac{|E_1|}{|\Omega|} = \dfrac{(\#\text{ outcomes in } E_1)}{(\#\text{ outcomes in } \Omega)} = \dfrac{4}{8} = \dfrac{1}{2} = \boxed{0.5}$

(c) Find the probability that Component 2 functions.

Let event $E_2 \equiv$ "Component 2 functions" $= \{FSF,\ FSS,\ SSF,\ SSS\}$

Then $\mathbb{P}(E_2) = \dfrac{|E_2|}{|\Omega|} = \dfrac{(\#\text{ outcomes in } E_2)}{(\#\text{ outcomes in } \Omega)} = \dfrac{4}{8} = \dfrac{1}{2} = \boxed{0.5}$

(d) Find the probability that the entire system functions.

Let event $G \equiv$ "Entire system functions" $= \{SSF,\ FFS,\ SFS,\ FSS,\ SSS\}$

Then $\mathbb{P}(G) = \dfrac{|G|}{|\Omega|} = \dfrac{(\#\text{ outcomes in } G)}{(\#\text{ outcomes in } \Omega)} = \dfrac{5}{8} = \boxed{0.625}$

(e) Find the probability that Component 1 functions given the entire system functions.

$$\mathbb{P}(E_1|G) = \frac{|E_1 \cap G|}{|G|} = \frac{(\#\text{ outcomes in } E_1 \cap G)}{(\#\text{ outcomes in } G)} = \frac{|\{SSF,\ SFS,\ SSS\}|}{|\{SSF,\ FFS,\ SFS,\ FSS,\ SSS\}|} = \frac{3}{5} = \boxed{0.60}$$

Notice that $\mathbb{P}(E_1|G) > \mathbb{P}(E_1)$, meaning if the entire system functions then it's more likely that Component 1 functions.

(f) Find the probability that if Component 2 functions then Component 1 functions.

$$\mathbb{P}(E_1|E_2) = \frac{|E_1 \cap E_2|}{|E_2|} = \frac{(\#\text{ outcomes in } E_1 \cap E_2)}{(\#\text{ outcomes in } E_2)} = \frac{|\{SSF,\ SSS\}|}{|\{FSF,\ FSS,\ SSF,\ SSS\}|} = \frac{2}{4} = \boxed{0.50}$$

Notice that $\mathbb{P}(E_1|E_2) = \mathbb{P}(E_1)$, meaning Component 2's working status does not affect Component 1's working status.

(g) Find the probability that Components 1 & 2 both function given entire system functions.

$$\mathbb{P}[(E_1 \cap E_2) \mid G] = \frac{|(E_1 \cap E_2) \cap G|}{|G|} = \frac{|\{SSF,\ SSS\}|}{|\{SSF,\ FFS,\ SFS,\ FSS,\ SSS\}|} = \frac{2}{5} = \boxed{0.40}$$

(h) Find the probability that Component 1 functions or Component 2 functions given entire system functions.

$$\mathbb{P}[(E_1 \cup E_2) \mid G] = \frac{|(E_1 \cup E_2) \cap G|}{|G|} = \frac{|\{FSF,\ FSS,\ SFF,\ SSF,\ SFS,\ SSS\} \cap \{SSF,\ FFS,\ SFS,\ FSS,\ SSS\}|}{|\{SSF,\ FFS,\ SFS,\ FSS,\ SSS\}|}$$

$$= \frac{|\{SSF,\ SFS,\ FSS,\ SSS\}|}{|\{SSF,\ FFS,\ SFS,\ FSS,\ SSS\}|} = \frac{4}{5} = \boxed{0.80}$$

EX 2.4.2: The incidence of a virus in a village is 10% and that a test correctly identifies the virus 95% of the time. Assume that false positives occur 8% of the time.

(a) Identify the events & probabilities that are given.

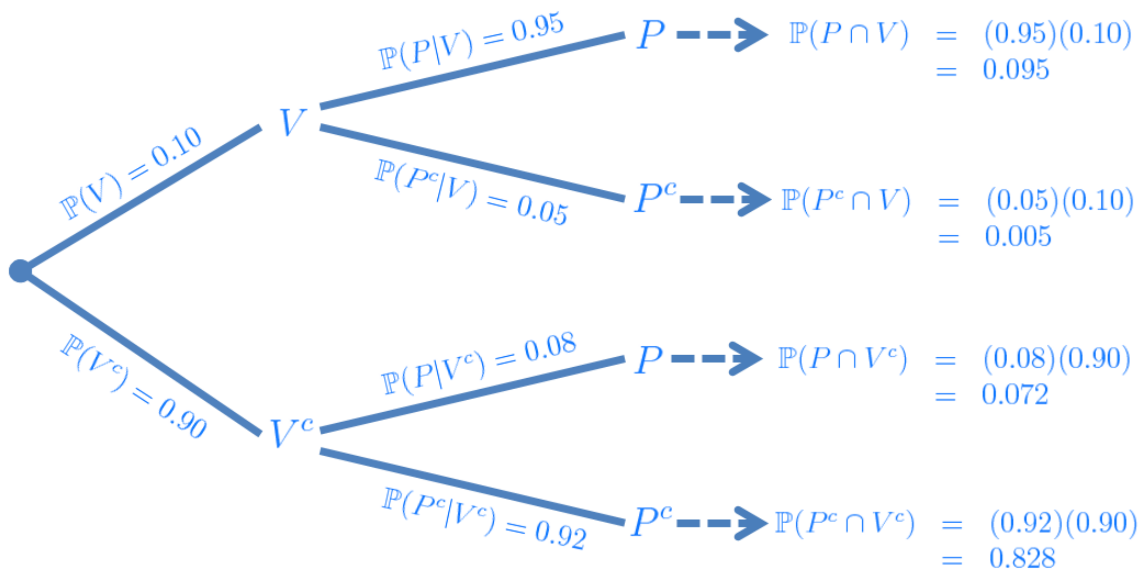
$V \equiv$ (Person has virus), $P \equiv$ (Person tests positive for virus)

$\mathbb{P}(V) = 0.10$, $\mathbb{P}(P|V) = 0.95$, $\mathbb{P}(P|V^c) = 0.08$

(b) Construct a probability tree representing the experiment. Label all relevant events & probabilities.

This can be viewed as a **2-stage experiment**:

Person reports whether he or she has virus, and then the person is tested for the virus.



(c) What is the probability of a person in the village having the virus?

$\mathbb{P}(V) = \boxed{0.10}$

(d) What is the probability of a person testing positive for the virus given the person has the virus?

$\mathbb{P}(P|V) = \boxed{0.95}$

(e) What is the probability that if a person has the virus then the person tests negative for the virus?

$\mathbb{P}(P^c|V) = \boxed{0.05}$

(f) What is the probability that a person has the virus and tests negative for the virus?

$\mathbb{P}(V \cap P^c) = \mathbb{P}(P^c \cap V) = \boxed{0.005}$

(g) What is the probability that a person has the virus given the person tests positive for the virus?

$$\mathbb{P}(V|P) = \frac{\mathbb{P}(V \cap P)}{\mathbb{P}(P)} \overset{LTP}{=} \frac{\mathbb{P}(P \cap V)}{\mathbb{P}(P \cap V) + \mathbb{P}(P \cap V^c)} = \frac{0.095}{0.095 + 0.072} = \frac{0.095}{0.167} = \frac{95}{167} \approx \boxed{0.5689}$$

Notice that the Law of Total Probability (LTP) was invoked, but overall Bayes' Theorem was implicitly invoked.

With a probability tree constructed, it's easier to use Law of Total Probability instead of Bayes' Theorem directly.

Moreover, notice how easier it is to directly work with **intersection probabilities** instead of **conditional probabilities**.

(h) What is the probability that a person does not have the virus given the person tests negative for the virus?

$$\mathbb{P}(V^c|P^c) = \frac{\mathbb{P}(V^c \cap P^c)}{\mathbb{P}(P^c)} \overset{LTP}{=} \frac{\mathbb{P}(P^c \cap V^c)}{\mathbb{P}(P^c \cap V) + \mathbb{P}(P^c \cap V^c)} = \frac{0.828}{0.005 + 0.828} = \frac{0.828}{0.833} = \frac{828}{833} \approx \boxed{0.9940}$$

EX 2.4.3: Two hundred engineering students were randomly selected and their height & eye color were recorded and tallied:

8% of the short students have blue eyes	13.5% of the tall students have blue eyes
11.5% of the short students have brown eyes	10.5% of the tall students have brown eyes
9% of the short students have green eyes	22% of the tall students have green eyes
15.5% of the short students have hazel eyes	10% of the tall students have hazel eyes

(a) Construct the joint probability table, label all relevant events & probabilities, including the column totals & row totals.

This can be viewed as a **2-stage experiment**: Assess a student's height, and then observe the student's eye color.

A 2-stage experiment results in a 2-way joint probability table where each non-total table entry is an **intersection probability**:

	(**Bl**)ue Eyes	(**Br**)own Eyes	(**G**)reen Eyes	(**H**)azel Eyes	**TOTAL**
(**S**)hort	$\mathbb{P}(S \cap Bl) = 0.08$	$\mathbb{P}(S \cap Br) = 0.115$	$\mathbb{P}(S \cap G) = 0.09$	$\mathbb{P}(S \cap H) = 0.155$	$\mathbb{P}(S) \overset{LTP}{=} 0.44$
(**T**)all	$\mathbb{P}(T \cap Bl) = 0.135$	$\mathbb{P}(T \cap Br) = 0.105$	$\mathbb{P}(T \cap G) = 0.22$	$\mathbb{P}(T \cap H) = 0.10$	$\mathbb{P}(T) \overset{LTP}{=} 0.56$
TOTAL	$\mathbb{P}(Bl) \overset{LTP}{=} 0.215$	$\mathbb{P}(Br) \overset{LTP}{=} 0.22$	$\mathbb{P}(G) \overset{LTP}{=} 0.31$	$\mathbb{P}(H) \overset{LTP}{=} 0.255$	**(DON'T CARE)**

Realize that the heights (S, T) partition sample space Ω. Moreover, the eye colors (Bl, Br, G, H) partition sample space Ω.

TOTAL entries are computed via Law of Total Probability:
$$\mathbb{P}(G) \overset{LTP}{=} \mathbb{P}(G \cap S) + \mathbb{P}(G \cap T) = 0.09 + 0.22 = 0.31$$
$$\mathbb{P}(T) \overset{LTP}{=} \mathbb{P}(T \cap Bl) + \mathbb{P}(T \cap Br) + \mathbb{P}(T \cap G) + \mathbb{P}(T \cap H)$$

(b) Find the probability that a student is tall and have brown eyes.

$\mathbb{P}(T \cap Br) = \boxed{0.105}$

(c) Find the probability that a student is short.

$\mathbb{P}(S) = \boxed{0.44}$

(d) Find the probability that a student has hazel eyes.

$\mathbb{P}(H) = \boxed{0.255}$

(e) Find the probability that a student has green eyes given the student is tall.

$$\mathbb{P}(G|T) = \frac{\mathbb{P}(G \cap T)}{\mathbb{P}(T)} = \frac{0.22}{0.56} = \frac{22}{56} \approx \boxed{0.3929}$$

(f) Find the probability that if a student has blue eyes then the student is short.

$$\mathbb{P}(S|Bl) = \frac{\mathbb{P}(S \cap Bl)}{\mathbb{P}(Bl)} = \frac{0.08}{0.215} = \frac{80}{215} \approx \boxed{0.3721}$$

(g) Find the probability that a student has blue eyes or hazel eyes.

$$\mathbb{P}(Bl \cup H) = \mathbb{P}(Bl) + \mathbb{P}(H) - \mathbb{P}(Bl \cap H) = 0.215 + 0.255 - 0 = \boxed{0.47}$$

Note that $\mathbb{P}(Bl \cap H) = 0$ since the eye colors are pairwise disjoint.

(h) Find the probability that a student has green eyes or hazel eyes given the student is tall.

$$\mathbb{P}[(G \cup H) \mid T] = \frac{\mathbb{P}[(G \cup H) \cap T]}{\mathbb{P}(T)} \overset{S10}{=} \frac{\mathbb{P}[(G \cap T) \cup (H \cap T)]}{\mathbb{P}(T)} = \frac{\mathbb{P}(G \cap T) + \mathbb{P}(H \cap T)}{\mathbb{P}(T)} = \frac{0.22 + 0.10}{0.56} \approx \boxed{0.5714}$$

(i) Find the probability that a student is tall given the student has green eyes or hazel eyes.

$$\mathbb{P}[T \mid (G \cup H)] = \frac{\mathbb{P}[T \cap (G \cup H)]}{\mathbb{P}(G \cup H)} \overset{S10}{=} \frac{\mathbb{P}[(T \cap G) \cup (T \cap H)]}{\mathbb{P}(G \cup H)} = \frac{\mathbb{P}(T \cap G) + \mathbb{P}(T \cap H)}{\mathbb{P}(G) + \mathbb{P}(H)} = \frac{0.22 + 0.10}{0.31 + 0.255} \approx \boxed{0.5664}$$

(j) Find the probability that a student has green eyes or blue eyes or brown eyes given the student is short.

$$\mathbb{P}[(G \cup Bl \cup Br) \mid S] = \frac{\mathbb{P}[(G \cup Bl \cup Br) \cap S]}{\mathbb{P}(S)} \overset{S10}{=} \frac{\mathbb{P}[(G \cap S) \cup (Bl \cap S) \cup (Br \cap S)]}{\mathbb{P}(S)} = \frac{\mathbb{P}(G \cap S) + \mathbb{P}(Bl \cap S) + \mathbb{P}(Br \cap S)}{\mathbb{P}(S)}$$

$$= \frac{0.09 + 0.08 + 0.115}{0.44} = \frac{0.285}{0.44} = \frac{285}{440} \approx \boxed{0.6477}$$