PROBABILITY: CONDITIONING, BAYES' THEOREM [DEVORE 2.4]

- **CONDITIONAL PROBABILITY:** Let events E, F be events in the sample space Ω of an experiment. Then:

  The **conditional probability** of F **given** E, $\mathbb{P}(F|E)$, is the probability of event F assuming E has already occurred.

  The **conditional probability** of E **given** F, $\mathbb{P}(E|F)$, is the probability of event E assuming F has already occurred.

  WARNING: **Order matters**: in general, $\mathbb{P}(F|E) \neq \mathbb{P}(E|F)$

- **CONDITIONAL PROBABILITY:** Let events E, F be events in the sample space Ω such that $\mathbb{P}(E) > 0$. Then:

$$\mathbb{P}(\text{If } E \text{ then } F) = \mathbb{P}(F \text{ given } E) = \mathbb{P}(F|E) := \frac{|E \cap F|}{|E|} = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)}$$

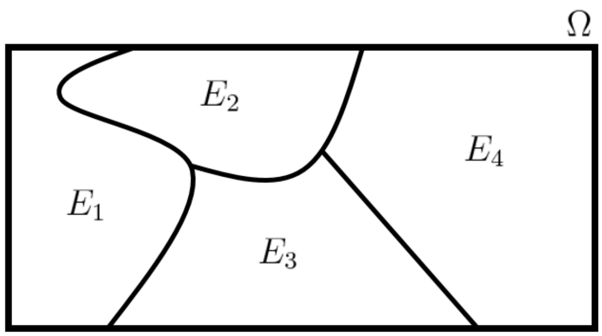
- **INTERSECTION OF EVENTS:** Let events E, F be events in the sample space Ω of an experiment. Then:

$$\mathbb{P}(E \cap F) = \mathbb{P}(E) \cdot \mathbb{P}(F|E) = \mathbb{P}(F) \cdot \mathbb{P}(E|F)$$

- **PARTITIONING A SAMPLE SPACE:** Events $E_1, E_2, \ldots, E_k \subseteq \Omega$ **partition** sample space Ω if:

$$E_1, E_2, \ldots, E_k \text{ are pairwise disjoint} \quad \text{AND} \quad \bigcup_{i=1}^{k} E_i = \Omega$$

  Think of the sample space Ω as a puzzle & the partitioning events $E_1, E_2, \ldots, E_k$ as the puzzle pieces.

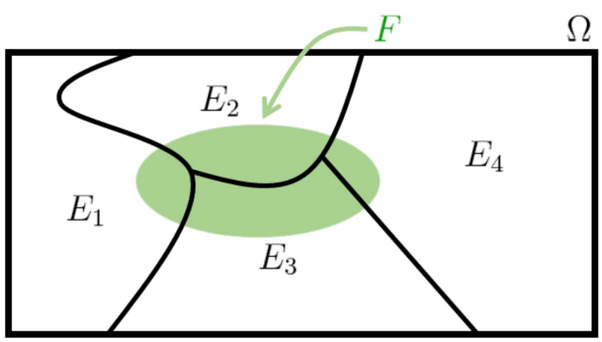


Events E_1, E_2, E_3, E_4 partition sample space Ω.

- **LAW OF TOTAL PROBABILITY (LTP):** Let events $E_1, \ldots, E_k$ partition sample space Ω. Then

$$\mathbb{P}(F) = \sum_{i=1}^{k} \mathbb{P}(F \cap E_i) = \mathbb{P}(F|E_i) \cdot \mathbb{P}(E_i)$$

  In practice, it's less tedious to work with the middle form involving intersections rather than right-most form.



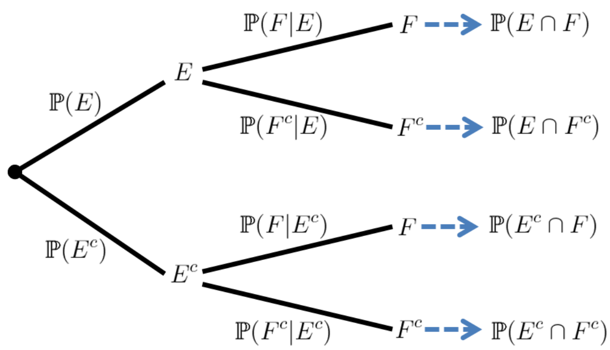
- **BAYES' THEOREM:** Let events $E_1, \ldots, E_k \subseteq \Omega$ partition sample space Ω. Then

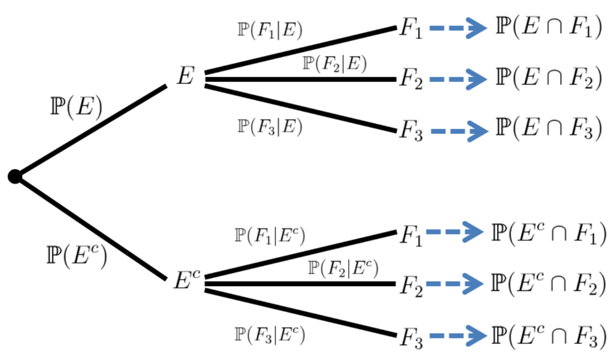
$$\mathbb{P}(E_j|F) = \frac{\mathbb{P}(F \cap E_j)}{\sum_{i=1}^{k} \mathbb{P}(F \cap E_i)} = \frac{\mathbb{P}(F|E_j) \cdot \mathbb{P}(E_j)}{\sum_{i=1}^{k} \mathbb{P}(F|E_i) \cdot \mathbb{P}(E_i)} \quad \text{for} \quad j = 1, 2, \ldots, k$$

  In practice, it's less tedious to work with the middle form involving intersections rather than right-most form.

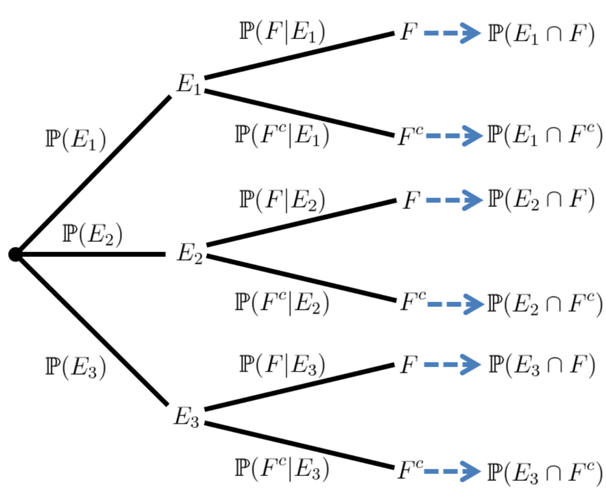
  Moreover, it's often far easier to directly work with the Law of Total Probability than directly with Bayes' Theorem.

PROBABILITY: PROBABILITY TREES [DEVORE 2.4]





Events F_1, F_2, F_3 must partition the sample space Ω.



Events E_1, E_2, E_3 must partition the sample space Ω.

PROBABILITY: JOINT PROBABILITY TABLES [DEVORE 2.4]

2-STAGE EXPERIMENTS & 2-WAY JOINT PROBABILITY TABLES:

	F	F^c	**TOTAL**
E	$\mathbb{P}(E \cap F)$	$\mathbb{P}(E \cap F^c)$	$\mathbb{P}(E)$
E^c	$\mathbb{P}(E^c \cap F)$	$\mathbb{P}(E^c \cap F^c)$	$\mathbb{P}(E^c)$
TOTAL	$\mathbb{P}(F)$	$\mathbb{P}(F^c)$	(**DON'T CARE**)

	F_1	F_2	F_2	**TOTAL**
E_1	$\mathbb{P}(E_1 \cap F_1)$	$\mathbb{P}(E_1 \cap F_2)$	$\mathbb{P}(E_1 \cap F_3)$	$\mathbb{P}(E_1)$
E_2	$\mathbb{P}(E_2 \cap F_1)$	$\mathbb{P}(E_2 \cap F_2)$	$\mathbb{P}(E_2 \cap F_3)$	$\mathbb{P}(E_2)$
E_3	$\mathbb{P}(E_3 \cap F_1)$	$\mathbb{P}(E_3 \cap F_2)$	$\mathbb{P}(E_3 \cap F_3)$	$\mathbb{P}(E_3)$
TOTAL	$\mathbb{P}(F_1)$	$\mathbb{P}(F_2)$	$\mathbb{P}(F_3)$	(**DON'T CARE**)

Events E_1, E_2, E_3 must partition sample space Ω.

Events F_1, F_2, F_3 must partition sample space Ω.

3-STAGE EXPERIMENTS & 3-WAY JOINT PROBABILITY TABLES:

E	G	G^c	**TOTAL**
F	$\mathbb{P}(E \cap F \cap G)$	$\mathbb{P}(E \cap F \cap G^c)$	$\mathbb{P}(E \cap F)$
F^c	$\mathbb{P}(E \cap F^c \cap G)$	$\mathbb{P}(E \cap F^c \cap G^c)$	$\mathbb{P}(E \cap F^c)$
TOTAL	$\mathbb{P}(E \cap G)$	$\mathbb{P}(E \cap G^c)$	(**DON'T CARE**)

E^c	G	G^c	**TOTAL**
F	$\mathbb{P}(E^c \cap F \cap G)$	$\mathbb{P}(E^c \cap F \cap G^c)$	$\mathbb{P}(E^c \cap F)$
F^c	$\mathbb{P}(E^c \cap F^c \cap G)$	$\mathbb{P}(E^c \cap F^c \cap G^c)$	$\mathbb{P}(E^c \cap F^c)$
TOTAL	$\mathbb{P}(E^c \cap G)$	$\mathbb{P}(E^c \cap G^c)$	(**DON'T CARE**)

E_1	G_1	G_2	G_3	**TOTAL**
F_1	$\mathbb{P}(E_1 \cap F_1 \cap G_1)$	$\mathbb{P}(E_1 \cap F_1 \cap G_2)$	$\mathbb{P}(E_1 \cap F_1 \cap G_3)$	$\mathbb{P}(E_1 \cap F_1)$
F_2	$\mathbb{P}(E_1 \cap F_2 \cap G_1)$	$\mathbb{P}(E_1 \cap F_2 \cap G_2)$	$\mathbb{P}(E_1 \cap F_2 \cap G_3)$	$\mathbb{P}(E_1 \cap F_2)$
F_3	$\mathbb{P}(E_1 \cap F_3 \cap G_1)$	$\mathbb{P}(E_1 \cap F_3 \cap G_2)$	$\mathbb{P}(E_1 \cap F_3 \cap G_3)$	$\mathbb{P}(E_1 \cap F_3)$
TOTAL	$\mathbb{P}(E_1 \cap G_1)$	$\mathbb{P}(E_1 \cap G_2)$	$\mathbb{P}(E_1 \cap G_3)$	(**DON'T CARE**)

E_2	G_1	G_2	G_3	**TOTAL**
F_1	$\mathbb{P}(E_2 \cap F_1 \cap G_1)$	$\mathbb{P}(E_2 \cap F_1 \cap G_2)$	$\mathbb{P}(E_2 \cap F_1 \cap G_3)$	$\mathbb{P}(E_2 \cap F_1)$
F_2	$\mathbb{P}(E_2 \cap F_2 \cap G_1)$	$\mathbb{P}(E_2 \cap F_2 \cap G_2)$	$\mathbb{P}(E_2 \cap F_2 \cap G_3)$	$\mathbb{P}(E_2 \cap F_2)$
F_3	$\mathbb{P}(E_2 \cap F_3 \cap G_1)$	$\mathbb{P}(E_2 \cap F_3 \cap G_2)$	$\mathbb{P}(E_2 \cap F_3 \cap G_3)$	$\mathbb{P}(E_2 \cap F_3)$
TOTAL	$\mathbb{P}(E_2 \cap G_1)$	$\mathbb{P}(E_2 \cap G_2)$	$\mathbb{P}(E_2 \cap G_3)$	(**DON'T CARE**)

E_3	G_1	G_2	G_3	**TOTAL**
F_1	$\mathbb{P}(E_3 \cap F_1 \cap G_1)$	$\mathbb{P}(E_3 \cap F_1 \cap G_2)$	$\mathbb{P}(E_3 \cap F_1 \cap G_3)$	$\mathbb{P}(E_3 \cap F_1)$
F_2	$\mathbb{P}(E_3 \cap F_2 \cap G_1)$	$\mathbb{P}(E_3 \cap F_2 \cap G_2)$	$\mathbb{P}(E_3 \cap F_2 \cap G_3)$	$\mathbb{P}(E_3 \cap F_2)$
F_3	$\mathbb{P}(E_3 \cap F_3 \cap G_1)$	$\mathbb{P}(E_3 \cap F_3 \cap G_2)$	$\mathbb{P}(E_3 \cap F_3 \cap G_3)$	$\mathbb{P}(E_3 \cap F_3)$
TOTAL	$\mathbb{P}(E_3 \cap G_1)$	$\mathbb{P}(E_3 \cap G_2)$	$\mathbb{P}(E_3 \cap G_3)$	(**DON'T CARE**)

Events E_1, E_2, E_3 must partition sample space Ω.

Events F_1, F_2, F_3 must partition sample space Ω.

Events G_1, G_2, G_3 must partition sample space Ω.

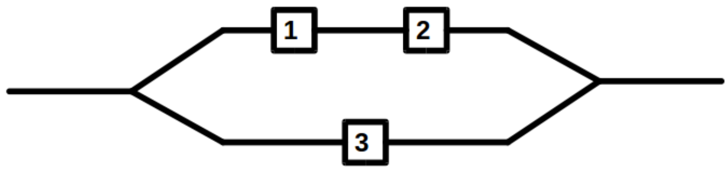
EX 2.4.1: Consider the following system of connected components shown below.

Components 1 & 2 are connected in series with component 3 connected to them in parallel.

For the entire system to function, either components 1 & 2 must function or component 3 must function.

The experiment consists of observing each component's condition and labeling it as either (S)uccess or (F)ailure.

So, for example, the outcome for components 1 & 2 functioning but 3 not functioning would be denoted as SSF.



(a) Determine the sample space Ω for the experiment.

(b) Find the probability that Component 1 functions.

(c) Find the probability that Component 2 functions.

(d) Find the probability that the entire system functions.

(e) Find the probability that Component 1 functions given the entire system functions. Interpret result in context.

(f) Find the probability that if Component 2 functions then Component 1 functions. Interpret result in context.

(g) Find the probability that Components 1 & 2 both function given entire system functions. Interpret result.

(h) Find the probability that Component 1 functions or Component 2 functions given entire system functions.

EX 2.4.2: The incidence of a virus in a village is 10% and that a test correctly identifies the virus 95% of the time. Assume that false positives occur 8% of the time.

(a) Identify the events & probabilities that are given.

(b) Construct a probability tree representing the experiment. Label all relevant events & probabilities.

(c) What is the probability of a person in the village having the virus?

(d) What is the probability of a person testing positive for the virus given the person has the virus?

(e) What is the probability that if a person has the virus then the person tests negative for the virus?

(f) What is the probability that a person has the virus and tests negative for the virus?

(g) What is the probability that a person has the virus given the person tests positive for the virus?

(h) What is the probability that a person does not have the virus given the person tests negative for the virus?

EX 2.4.3: Two hundred engineering students were randomly selected and their height & eye color were recorded and tallied:

8% of the short students have blue eyes	13.5% of the tall students have blue eyes
11.5% of the short students have brown eyes	10.5% of the tall students have brown eyes
9% of the short students have green eyes	22% of the tall students have green eyes
15.5% of the short students have hazel eyes	10% of the tall students have hazel eyes

(a) Construct the joint probability table, label all relevant events & probabilities, including the column totals & row totals.

(b) Find the probability that a student is tall and have brown eyes.

(c) Find the probability that a student is short.

(d) Find the probability that a student has hazel eyes.

(e) Find the probability that a student has green eyes given the student is tall.

(f) Find the probability that if a student has blue eyes then the student is short.

(g) Find the probability that a student has blue eyes or hazel eyes.

(h) Find the probability that a student has green eyes or hazel eyes given the student is tall.

(i) Find the probability that a student is tall given the student has green eyes or hazel eyes.

(j) Find the probability that a student has green eyes or blue eyes or brown eyes given the student is short.