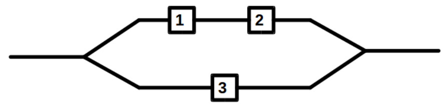
EX 2.5.1: Consider the following system of connected components shown below.

Components 1 & 2 are connected in series with component 3 connected to them in parallel.

For the entire system to function, either components 1 & 2 must function or component 3 must function.

The experiment consists of observing each component's condition and labeling it as either (S)uccess or (F)ailure.

So, for example, the outcome for components 1 & 2 functioning but 3 not functioning would be denoted as SSF.



(a) Determine the sample space Ω for the experiment.

$\Omega =$ (The set of all possible outcomes) $= \boxed{\{FFF,\ FFS,\ FSF,\ FSS,\ SFF,\ SSF,\ SFS,\ SSS\}}$

(b) Are the events "Component 1 functions" & "Component 2 functions" independent? (Justify answer.)

Let $A \equiv$ "Component 1 functions" $= \{SFF,\ SSF,\ SFS,\ SSS\} \implies \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$

Let $B \equiv$ "Component 2 functions" $= \{FSF,\ FSS,\ SSF,\ SSS\} \implies \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$

Then $A \cap B = \{SSF,\ SSS\} \implies \mathbb{P}(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{2}{8} = \frac{1}{4}$. Moreover, $\mathbb{P}(A) \cdot \mathbb{P}(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} = \mathbb{P}(A \cap B)$

$\therefore$ Since $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$, the two events are independent.

(c) Are the events "Component 1 functions" & "Exactly two components function" independent? (Justify answer.)

Let $D \equiv$ "Exactly two components function" $= \{FSS,\ SSF,\ SFS\} \implies \mathbb{P}(D) = \frac{|D|}{|\Omega|} = \frac{3}{8}$

Then $A \cap D = \{SSF,\ SFS\} \implies \mathbb{P}(A \cap D) = \frac{|A \cap D|}{|\Omega|} = \frac{2}{8} = \frac{1}{4}$. Moreover, $\mathbb{P}(A) \cdot \mathbb{P}(D) = \left(\frac{1}{2}\right)\left(\frac{3}{8}\right) = \frac{3}{16} \neq \mathbb{P}(A \cap D)$

$\therefore$ Since $\mathbb{P}(A \cap D) \neq \mathbb{P}(A) \cdot \mathbb{P}(D)$, the two events are not independent.

(d) Do the three components function independently of one another? (Justify answer.)

Let $C \equiv$ "Component 3 functions" $= \{FSS,\ FFS,\ SFS,\ SSS\} \implies \mathbb{P}(C) = \frac{|C|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$

$A \cap C = \{SFS,\ SSS\} \implies \mathbb{P}(A \cap C) = \frac{|A \cap C|}{|\Omega|} = \frac{2}{8} = \frac{1}{4}$

$B \cap C = \{FSS,\ SSS\} \implies \mathbb{P}(B \cap C) = \frac{|B \cap C|}{|\Omega|} = \frac{2}{8} = \frac{1}{4}$

$A \cap B \cap C = \{SSS\} \implies \mathbb{P}(A \cap B \cap C) = \frac{|A \cap B \cap C|}{|\Omega|} = \frac{1}{8}$

Therefore:

$$\mathbb{P}(A \cap B) = \frac{1}{4} \qquad \mathbb{P}(A) \cdot \mathbb{P}(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$\mathbb{P}(A \cap C) = \frac{1}{4} \qquad \mathbb{P}(A) \cdot \mathbb{P}(C) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

AND

$$\mathbb{P}(B \cap C) = \frac{1}{4} \qquad \mathbb{P}(B) \cdot \mathbb{P}(C) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$\mathbb{P}(A \cap B \cap C) = \frac{1}{8} \qquad \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

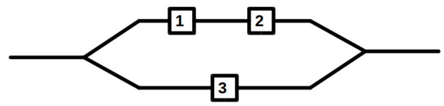
$\therefore$ Since $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$, $\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$, $\mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C)$, and $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$,

the three components do function independently of one another.

EX 2.5.2: Consider the following system of connected components shown below.

Components 1 & 2 are connected in series with component 3 connected to them in parallel.

For the entire system to function, either components 1 & 2 must function or component 3 must function.



Suppose the components work independently of one another. Moreover, suppose that:

Component 1 works 95% of the time, Component 2 works 97% of the time, and Component 3 works 91% of the time.

What is the probability that the entire system works?

Let $A \equiv$ "Component 1 works", $B \equiv$ "Component 2 works", $C \equiv$ "Component 3 works"

Then:

$$\mathbb{P}(A) = 0.95 \qquad \mathbb{P}(B) = 0.97 \qquad \mathbb{P}(C) = 0.91$$
$$\mathbb{P}(A^c) = 0.05 \qquad \mathbb{P}(B^c) = 0.03 \qquad \mathbb{P}(C^c) = 0.09$$

(IND) "The components work independently of one another" means events A, B, C are mutually independent.

$$\begin{aligned}
\therefore\ \mathbb{P}(\text{Entire System Works}) &= \mathbb{P}[(A\cap B\cap C)\cup(A^c\cap B^c\cap C)\cup(A\cap B^c\cap C)\cup(A^c\cap B\cap C)\cup(A\cap B\cap C^c)]\\
&\overset{PD}{=} \mathbb{P}(A\cap B\cap C)+\mathbb{P}(A^c\cap B^c\cap C)+\mathbb{P}(A\cap B^c\cap C)+\mathbb{P}(A^c\cap B\cap C)+\mathbb{P}(A\cap B\cap C^c)\\
&\overset{IND}{=} \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)+\mathbb{P}(A^c)\mathbb{P}(B^c)\mathbb{P}(C)+\mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C)+\mathbb{P}(A^c)\mathbb{P}(B)\mathbb{P}(C)+\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C^c)\\
&= (0.95)(0.97)(0.91)+(0.05)(0.03)(0.91)+(0.95)(0.03)(0.91)+(0.05)(0.97)(0.91)+(0.95)(0.97)(0.09)\\
&= \boxed{\mathbf{0.992935}}
\end{aligned}$$

ALTERNATIVELY

$$\begin{aligned}
\therefore\ \mathbb{P}(\text{Entire System Works}) &= 1-\mathbb{P}(\text{Entire System does } \underline{\text{not}} \text{ Work})\\
&= 1-\mathbb{P}[(A^c\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c)\cup(A\cap B^c\cap C^c)]\\
&\overset{PD}{=} 1-[\mathbb{P}(A^c\cap B^c\cap C^c)+\mathbb{P}(A^c\cap B\cap C^c)+\mathbb{P}(A\cap B^c\cap C^c)]\\
&\overset{IND}{=} 1-[\mathbb{P}(A^c)\mathbb{P}(B^c)\mathbb{P}(C^c)+\mathbb{P}(A^c)\mathbb{P}(B)\mathbb{P}(C^c)+\mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C^c)]\\
&= 1-[(0.05)(0.03)(0.09)+(0.05)(0.97)(0.09)+(0.95)(0.03)(0.09)]\\
&= 1-0.007065\\
&= \boxed{\mathbf{0.992935}}
\end{aligned}$$

(The moral here is that sometimes it's less tedious to work with the complement.)