EX 3.2.1: Consider the following experiment: Flip two fair coins and observe their top faces.

Let random variable $X \equiv$ (# Heads Observed)

Let random variable $Z \equiv$ (Is at least One Tail Observed? $(1 = \text{Yes},\ 0 = \text{No})$)

(a) List all the possible outcomes in the sample space Ω for the experiment.

$$\Omega = \boxed{\{HH,\ HT,\ TH,\ TT\}}$$

(b) For each outcome in the sample space Ω, determine the associated value of each random variable X, Z.

$$X(HH) = 2, \quad X(HT) = 1, \quad X(TH) = 1, \quad X(TT) = 0$$

$$Z(HH) = 0, \quad Z(HT) = 1, \quad Z(TH) = 1, \quad Z(TT) = 1$$

(c) Determine the support of each random variable X, Z for the experiment.

$$\text{Supp}(X) = \boxed{\{0, 1, 2\}} \qquad \text{Supp}(Z) = \boxed{\{0, 1\}}$$

(d) Determine the probability mass function (pmf) for each random variable X, Z.

$$\begin{cases} p_X(0) = \mathbb{P}(X = 0) = \dfrac{|\{TT\}|}{|\Omega|} = \dfrac{1}{4} \\ p_X(1) = \mathbb{P}(X = 1) = \dfrac{|\{HT,\ TH\}|}{|\Omega|} = \dfrac{1}{2} \\ p_X(2) = \mathbb{P}(X = 2) = \dfrac{|\{HH\}|}{|\Omega|} = \dfrac{1}{4} \end{cases} \implies$$

The pmf of r.v. X is

k	0	1	2
$p_X(k)$	1/4	1/2	1/4

$$\begin{cases} p_Z(0) = \mathbb{P}(Z = 0) = \dfrac{|\{HH\}|}{|\Omega|} = \dfrac{1}{4} \\ p_Z(1) = \mathbb{P}(Z = 1) = \dfrac{|\{HT,\ TH,\ TT\}|}{|\Omega|} = \dfrac{3}{4} \end{cases} \implies$$

The pmf of r.v. Z is

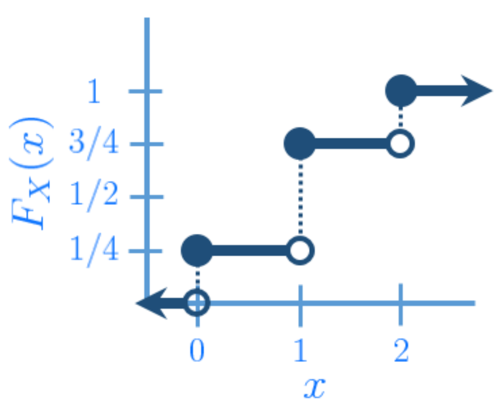
k	0	1
$p_Z(k)$	1/4	3/4

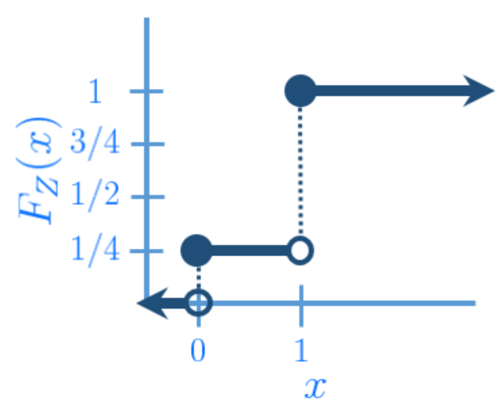
(e) Determine the cumulative distribution function (cdf) for each random variable X, Z.

$$p_X(k) = \begin{cases} 1/4 & \text{, if } k = 0 \\ 1/2 & \text{, if } k = 1 \\ 1/4 & \text{, if } k = 2 \end{cases} \implies F_X(x) = \begin{cases} 0 & \text{, if } x < 0 \\ \frac{1}{4} & \text{, if } 0 \leq x < 1 \\ \frac{1}{4} + \frac{1}{2} & \text{, if } 1 \leq x < 2 \\ \frac{1}{4} + \frac{1}{2} + \frac{1}{4} & \text{, if } 2 \leq x \end{cases} \implies \boxed{\text{cdf of } X \text{ is } F_X(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1/4 & \text{, if } 0 \leq x < 1 \\ 3/4 & \text{, if } 1 \leq x < 2 \\ 1 & \text{, if } 2 \leq x \end{cases}}$$

$$p_Z(k) = \begin{cases} 1/4 & \text{, if } k = 0 \\ 3/4 & \text{, if } k = 1 \end{cases} \implies F_Z(x) = \begin{cases} 0 & \text{, if } x < 0 \\ \frac{1}{4} & \text{, if } 0 \leq x < 1 \\ \frac{1}{4} + \frac{3}{4} & \text{, if } 1 \leq x \end{cases} \implies \boxed{\text{cdf of } Z \text{ is } F_Z(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1/4 & \text{, if } 0 \leq x < 1 \\ 1 & \text{, if } 1 \leq x \end{cases}}$$

(f) Sketch the cdf for each random variable.





EX 3.2.2: Consider the following experiment: Repeatedly flip a fair coin and observe its top face until a tail occurs. Moreover, the coins flips are independent of one another.

Let random variable $X \equiv$ (# Heads Observed)
Let random variable $Z \equiv$ (Is at least One Tail Observed? (1 = Yes, 0 = No))

(a) List four possible outcomes in the sample space Ω for the experiment.

$$\Omega = \boxed{\{T,\ HT,\ HHT,\ HHHT,\ \cdots\}}$$

(b) For the four outcomes in the sample space Ω listed in part (a), determine the associated value of each rv X, Z.

$$X(T) = 0, \quad X(HT) = 1, \quad X(HHT) = 2, \quad X(HHHT) = 3, \quad \cdots$$
$$Z(T) = 1, \quad Z(HT) = 1, \quad Z(HHT) = 1, \quad Z(HHHT) = 1, \quad \cdots$$

(c) Determine the support of each random variable X, Z for the experiment.

$$\text{Supp}(X) = \boxed{\{0, 1, 2, 3, \cdots\}} \qquad \text{Supp}(Z) = \boxed{\{1\}}$$

(d) Determine the probability mass function (pmf) for each random variable X, Z.

Since the coin is fair, $\mathbb{P}(\text{Heads}) = \mathbb{P}(H) = 1/2$ and $\mathbb{P}(\text{Tails}) = \mathbb{P}(T) = 1/2$

$$\begin{array}{lllllllllllll}
p_X(0) & = & \mathbb{P}(X=0) & = & \mathbb{P}(T) & = & & & & & & & 1/2 \\
p_X(1) & = & \mathbb{P}(X=1) & = & \mathbb{P}(HT) & = & \mathbb{P}(H \cap T) & \stackrel{IND}{=} & \mathbb{P}(H)\cdot\mathbb{P}(T) & = & \frac{1}{2}\cdot\frac{1}{2} & = & 1/4 \\
p_X(2) & = & \mathbb{P}(X=2) & = & \mathbb{P}(HHT) & = & \mathbb{P}(H \cap H \cap T) & \stackrel{IND}{=} & \mathbb{P}(H)\cdot\mathbb{P}(H)\cdot\mathbb{P}(T) & = & \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} & = & 1/8 \\
p_X(3) & = & \mathbb{P}(X=3) & = & \mathbb{P}(HHHT) & = & \mathbb{P}(H \cap H \cap H \cap T) & \stackrel{IND}{=} & \mathbb{P}(H)\cdot\mathbb{P}(H)\cdot\mathbb{P}(H)\cdot\mathbb{P}(T) & = & \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} & = & 1/16 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

$\therefore$ The pmf of r.v. X is

k	0	1	2	3	$\cdots$
$p_X(k)$	1/2	1/4	1/8	1/16	$\cdots$

Since Supp(Z) contains exactly one value, its probability has to be one:
$p_Z(0) = \mathbb{P}(Z = 0) = 0$
$p_Z(1) = \mathbb{P}(Z = 1) = 1$

$\therefore$ The pmf of r.v. Z is

k	0	1
$p_Z(k)$	0	1

(e) Write the pmf of random variable X in closed-form by means of pattern recognition.

It's easier to recognize a pattern in the intermediate expression for each probability instead of its final value:

$$\begin{array}{lllll}
p_X(0) & = & \frac{1}{2} & = & \left(\frac{1}{2}\right)^1 \\
p_X(1) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \left(\frac{1}{2}\right)^2 \\
p_X(2) & = & \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} & = & \left(\frac{1}{2}\right)^3 \\
p_X(3) & = & \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} & = & \left(\frac{1}{2}\right)^4 \\
\vdots & & \vdots & & \vdots
\end{array}$$

$\therefore$ The closed-form formula for the pmf of r.v. X is $\boxed{p_X(k) = \left(\frac{1}{2}\right)^{k+1}}$ or $\boxed{p_X(k) = 0.5^{k+1}}$

SANITY CHECK: $\displaystyle\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1} = \sum_{k=0}^{\infty}\frac{1}{2}\cdot\left(\frac{1}{2}\right)^k = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \stackrel{(*)}{=} \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{2}\cdot 2 = 1$ ✓

$(*)$ Recall from Calculus II the sum of a convergent **geometric series**: If $a \neq 0$ and $|r| < 1$, then $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

EX 3.2.3: Consider the following experiment: Repeatedly flip an unfair coin and observe its top face until a tail occurs.

The coin is unfair because tails occurs twice as often as heads.

Moreover, the coins flips are independent of one another.

Let random variable $X \equiv$ (# Heads Observed)
Let random variable $Z \equiv$ (Is at least One Tail Observed? $(1 = \text{Yes},\ 0 = \text{No})$)

(a) List four possible outcomes in the sample space Ω for the experiment.

$$\Omega = \boxed{\{T,\ HT,\ HHT,\ HHHT,\ \cdots\}}$$

(b) For the four outcomes in the sample space Ω listed in part (a), determine the associated value of each rv X, Z.

$$X(T) = 0, \quad X(HT) = 1, \quad X(HHT) = 2, \quad X(HHHT) = 3, \quad \cdots$$
$$Z(T) = 1, \quad Z(HT) = 1, \quad Z(HHT) = 1, \quad Z(HHHT) = 1, \quad \cdots$$

(c) Determine the support of each random variable X, Z for the experiment.

$$\text{Supp}(X) = \boxed{\{0, 1, 2, 3, \cdots\}} \qquad \text{Supp}(Z) = \boxed{\{1\}}$$

(d) Determine the probability mass function (pmf) for each random variable X, Z.

Since the coin is unfair and tails occurs twice as often as heads:

$$\begin{cases} \mathbb{P}(T) = 2 \cdot \mathbb{P}(H) \\ \mathbb{P}(H) + \mathbb{P}(T) = 1 \end{cases} \implies \mathbb{P}(H) + 2 \cdot \mathbb{P}(H) = 1 \implies 3 \cdot \mathbb{P}(H) = 1 \implies \mathbb{P}(H) = 1/3 \implies \mathbb{P}(T) = 2 \cdot \mathbb{P}(H) = 2 \cdot \tfrac{1}{3} = 2/3$$

$$\begin{array}{lllllll}
p_X(0) = \mathbb{P}(X=0) = \mathbb{P}(T) = & & & & & & 2/3 \\
p_X(1) = \mathbb{P}(X=1) = \mathbb{P}(HT) = & \mathbb{P}(H \cap T) & \overset{IND}{=} & \mathbb{P}(H) \cdot \mathbb{P}(T) & = & \frac{1}{3} \cdot \frac{2}{3} & = 2/9 \\
p_X(2) = \mathbb{P}(X=2) = \mathbb{P}(HHT) = & \mathbb{P}(H \cap H \cap T) & \overset{IND}{=} & \mathbb{P}(H) \cdot \mathbb{P}(H) \cdot \mathbb{P}(T) & = & \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} & = 2/27 \\
p_X(3) = \mathbb{P}(X=3) = \mathbb{P}(HHHT) = & \mathbb{P}(H \cap H \cap H \cap T) & \overset{IND}{=} & \mathbb{P}(H) \cdot \mathbb{P}(H) \cdot \mathbb{P}(H) \cdot \mathbb{P}(T) & = & \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} & = 2/81 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots
\end{array}$$

$\therefore$ The pmf of r.v. X is

k	0	1	2	3	$\cdots$
$p_X(k)$	2/3	2/9	2/27	2/81	$\cdots$

Since Supp(Z) contains exactly one value, its probability has to be one: $p_Z(0) = \mathbb{P}(Z=0) = 0$, $p_Z(1) = \mathbb{P}(Z=1) = 1$ $\implies$

k	0	1
$p_Z(k)$	0	1

(e) Write the pmf of random variable X in closed-form by means of pattern recognition.

It's easier to recognize a pattern in the intermediate expression for each probability instead of its final value:

$$\begin{array}{lcl}
p_X(0) = \frac{2}{3} & = & \frac{2}{3} \cdot \left(\frac{1}{3}\right)^0 \\
p_X(1) = \frac{1}{3} \cdot \frac{2}{3} & = & \frac{2}{3} \cdot \left(\frac{1}{3}\right)^1 \\
p_X(2) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} & = & \frac{2}{3} \cdot \left(\frac{1}{3}\right)^2 \\
p_X(3) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} & = & \frac{2}{3} \cdot \left(\frac{1}{3}\right)^3 \\
\vdots & & \vdots
\end{array}$$

$\therefore$ The closed-form formula for the pmf of r.v. X is $\boxed{p_X(k) = \frac{2}{3} \cdot \left(\frac{1}{3}\right)^k}$

SANITY CHECK: $$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} \frac{2}{3} \cdot \left(\frac{1}{3}\right)^k = \frac{2}{3} \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \overset{(*)}{=} \frac{2}{3} \cdot \left(\frac{1}{1 - \frac{1}{3}}\right) = \frac{2}{3} \cdot \frac{3}{2} = 1 \ \checkmark$$

$(*)$ Recall from Calculus II the sum of a convergent **geometric series**: If $a \neq 0$ and $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

EX 3.2.4: Consider the following experiment: Flip two fair coins and observe their top faces. (pmf & cdf of X were found in EX 3.2.1)

Let random variable $X \equiv$ (# Heads Observed)

Compute the following probabilities in two ways, one way using the pmf of X and the second way using the cdf of X:

$\mathbb{P}(\text{Exactly one heads}) = \mathbb{P}(X=1) = p_X(1) = \boxed{1/2}$

$\mathbb{P}(\text{Exactly one heads}) = \mathbb{P}(X=1) = F_X(1) - F_X(1-) = F_X(1) - F_X(0) = 3/4 - 1/4 = \boxed{1/2}$

$\mathbb{P}(\text{At most one heads}) = \mathbb{P}(X \leq 1) = p_X(0) + p_X(1) = 1/4 + 1/2 = \boxed{3/4}$

$\mathbb{P}(\text{At most one heads}) = \mathbb{P}(X \leq 1) = F_X(1) = \boxed{3/4}$

$\mathbb{P}(\text{At least one heads}) = \mathbb{P}(X \geq 1) = p_X(1) + p_X(2) = 1/2 + 1/4 = \boxed{3/4}$

$\mathbb{P}(\text{At least one heads}) = \mathbb{P}(X \geq 1) = 1 - F_X(1-) = 1 - F_X(0) = 1 - 1/4 = \boxed{3/4}$

$\mathbb{P}(\text{Either no heads or two heads}) = \mathbb{P}(X=0 \text{ or } X=2) = p_X(0) + p_X(2) = 1/4 + 1/4 = \boxed{1/2}$

$\mathbb{P}(\text{Either no heads or two heads}) = \mathbb{P}(X=0 \text{ or } X=2) = [F_X(0) - F_X(0-)] + [F_X(2) - F_X(2-)] = [1/4 - 0] + [1 - 3/4] = \boxed{1/2}$

EX 3.2.5: Consider the following experiment: Repeatedly flip a fair coin and observe its top face until a tail occurs. Moreover, the coins flips are independent of one another.

Let random variable $X \equiv$ (# Heads Observed)

Compute the following probabilities using the pmf of X: (pmf of X, $p_X(k)$, was found in EX 3.2.2)

$\mathbb{P}(\text{Exactly one heads}) = \mathbb{P}(X=1) = p_X(1) = \left(\frac{1}{2}\right)^{1+1} = \left(\frac{1}{2}\right)^2 = \boxed{1/4}$

$\mathbb{P}(\text{At most one heads}) = \mathbb{P}(X \leq 1) = p_X(0) + p_X(1) = \left(\frac{1}{2}\right)^{0+1} + \left(\frac{1}{2}\right)^{1+1} = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \boxed{3/4}$

$\mathbb{P}(\text{At least one heads}) = \mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X=0) = 1 - p_X(0) = 1 - \left(\frac{1}{2}\right)^{0+1} = 1 - \frac{1}{2} = \boxed{1/2}$

$\mathbb{P}(\text{Between two heads and six heads, inclusive}) = \mathbb{P}(2 \leq X \leq 6) = p_X(2) + p_X(3) + p_X(4) + p_X(5) + p_X(6) = \boxed{31/128} \approx 0.2422$

$\mathbb{P}(\text{Between two heads and six heads, exclusive}) = \mathbb{P}(2 < X < 6) = p_X(3) + p_X(4) + p_X(5) = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 = \boxed{7/64} \approx 0.1094$

EX 3.2.6: Consider the following experiment: Repeatedly flip an unfair coin and observe its top face until a tail occurs. The coin is unfair because tails occurs twice as often as heads. Moreover, the coins flips are independent of one another.

Let random variable $X \equiv$ (# Heads Observed)

Compute the following probabilities using the pmf of X: (pmf of X, $p_X(k)$, was found in EX 3.2.3)

$\mathbb{P}(\text{Exactly one heads}) = \mathbb{P}(X=1) = p_X(1) = \frac{2}{3} \cdot \left(\frac{1}{3}\right)^1 = \frac{2}{3} \cdot \frac{1}{3} = \boxed{2/9}$

$\mathbb{P}(\text{At most one heads}) = \mathbb{P}(X \leq 1) = p_X(0) + p_X(1) = \frac{2}{3} \cdot \left(\frac{1}{3}\right)^0 + \frac{2}{3} \cdot \left(\frac{1}{3}\right)^1 = \frac{2}{3} + \frac{2}{9} = \boxed{8/9}$

$\mathbb{P}(\text{At least one heads}) = \mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X=0) = 1 - p_X(0) = 1 - \frac{2}{3} \cdot \left(\frac{1}{3}\right)^0 = 1 - \frac{2}{3} = \boxed{1/3}$

$\mathbb{P}(\text{Between two heads and six heads, inclusive}) = \mathbb{P}(2 \leq X \leq 6) = p_X(2) + p_X(3) + p_X(4) + p_X(5) + p_X(6) = \boxed{242/2187} \approx 0.1107$

$\mathbb{P}(\text{Between two heads and six heads, exclusive}) = \mathbb{P}(2 < X < 6) = p_X(3) + p_X(4) + p_X(5) = \frac{2}{3} \cdot \left(\frac{1}{3}\right)^3 + \frac{2}{3} \cdot \left(\frac{1}{3}\right)^4 + \frac{2}{3} \cdot \left(\frac{1}{3}\right)^5 = \boxed{26/729}$
