DISCRETE RANDOM VARIABLES: PMF's & CDF's [DEVORE 3.2]

- **PROBABILITY MASS FUNCTION (PMF) DEFINITION):** Let X be a **discrete** random variable.

  Then, its **pmf**, denoted as $p_X(k)$, is defined as follows: $p_X(k) := \mathbb{P}(X = k) \quad \forall k \in \text{Supp}(X)$

- **PROBABILITY MASS FUNCTION (PMF) AXIOMS):** The **pmf** $p_X(k)$ of a discrete r.v. X satisfies:

  Non-negativity on its Support: $p_X(k) \geq 0 \qquad \forall k \in \text{Supp}(X)$

  Universal Sum of Unity: $\displaystyle\sum_{k \in \text{Supp}(X)} p_X(k) = 1$

- **CUMULATIVE DENSITY FCN (CDF) DEFN:** Let X be a **discrete** random variable s.t. $\text{Supp}(X) = \{k_1, k_2, k_3, \cdots\}$

  Then, its **cdf**, denoted as $F_X(x)$, is defined as follows: $F_X(x) := \mathbb{P}(X \leq x) = \displaystyle\sum_{k_i \leq x} p_X(k_i) \quad \forall x \in \mathbb{R}$

- **CUMULATIVE DENSITY FCN (CDF) AXIOMS:** The **cdf** $F_X(x)$ of a discrete r.v. X satisfies:

  Eventually Zero (One) to the Left (Right): $\displaystyle\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to +\infty} F_X(x) = 1$

  Non-decreasing: $x_1 \leq x_2 \implies F_X(x_1) \leq F_X(x_2)$

  Right-continuous: $\displaystyle\lim_{x \downarrow x_0} F_X(x) = F_X(x_0) \quad \forall x_0 \in \mathbb{R}$

  Piecewise Constant: (AKA step function)

- **COMPUTING PROBABILITIES USING A DISCRETE PMF:** Let X be a **discrete** r.v. with cdf $F_X(x)$.

  Let scalars $a, b \in \mathbb{R}$ s.t. $a < b$. Then:

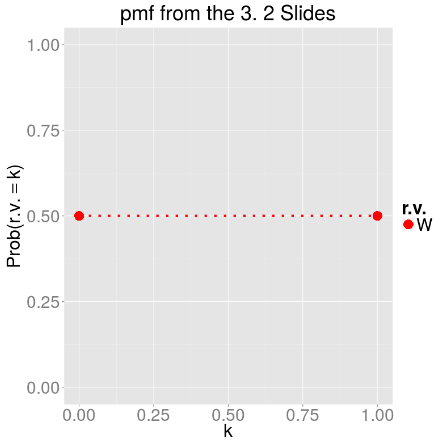
$$
\begin{aligned}
\mathbb{P}(X \leq a) &= F_X(a) \\
\mathbb{P}(X < a) &= F_X(a-) \\
\\
\mathbb{P}(a \leq X \leq b) &= F_X(b) \quad - F_X(a-) \\
\mathbb{P}(a < X < b) &= F_X(b-) - F_X(a) \\
\mathbb{P}(a < X \leq b) &= F_X(b) \quad - F_X(a) \\
\mathbb{P}(a \leq X < b) &= F_X(b-) - F_X(a-) \\
\\
\mathbb{P}(X \geq b) &= 1 - F_X(b-) \\
\mathbb{P}(X > b) &= 1 - F_X(b) \\
\\
\mathbb{P}(X = a) &= F_X(a) - F_X(a-)
\end{aligned}
$$

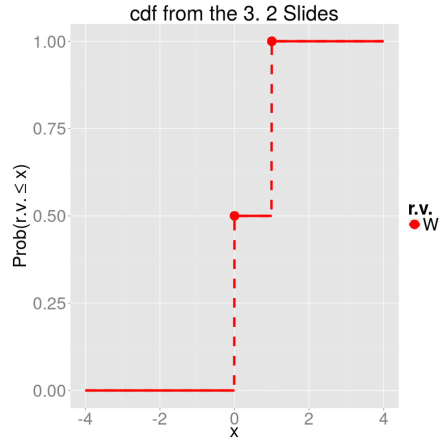
where: "$a-$" represents the **largest** $k \in \text{Supp}(X)$ such that $k < a$

"$b-$" represents the **largest** $k \in \text{Supp}(X)$ such that $k < b$

DISCRETE RANDOM VARIABLES: PMF & CDF PLOTS [DEVORE 3.2]

k	0	1
$p_W(k)$	$1/2$	$1/2$





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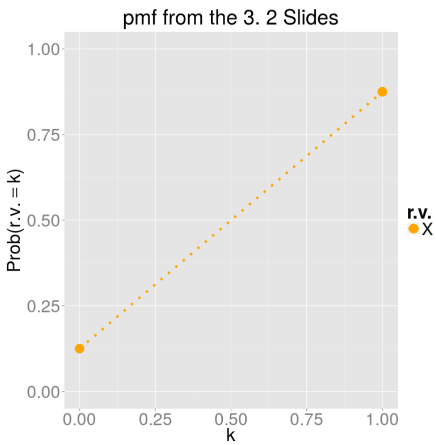
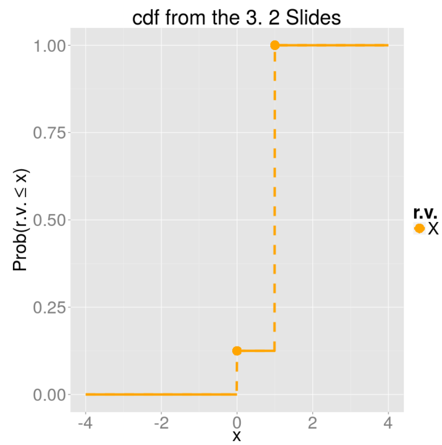
k	0	1
$p_X(k)$	$1/8$	$7/8$





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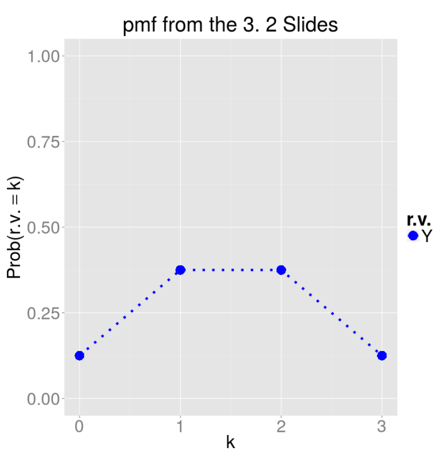
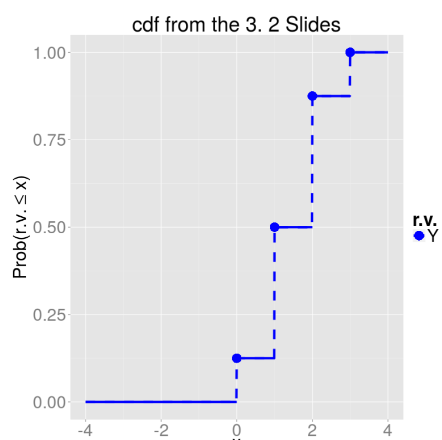
k	0	1	2	3
$p_Y(k)$	$1/8$	$3/8$	$3/8$	$1/8$





EX 3.2.1: Consider the following experiment: Flip two fair coins and observe their top faces.

Let random variable $X \equiv$ (# Heads Observed)
Let random variable $Z \equiv$ (Is at least One Tail Observed? (1 = Yes, 0 = No))

(a) List all the possible outcomes in the sample space Ω for the experiment.

(b) For each outcome in the sample space Ω, determine the associated value of each random variable X, Z.

(c) Determine the support of each random variable X, Z for the experiment.

(d) Determine the probability mass function (pmf) for each random variable X, Z.

(e) Determine the cumulative distribution function (cdf) for each random variable X, Z.

(f) Sketch the cdf for each random variable.

EX 3.2.2: Consider the following experiment: Repeatedly flip a fair coin and observe its top face until a tail occurs. Moreover, the coins flips are independent of one another.

Let random variable $X \equiv$ (# Heads Observed)
Let random variable $Z \equiv$ (Is at least One Tail Observed? $(1 = \text{Yes}, 0 = \text{No})$)

(a) List four possible outcomes in the sample space Ω for the experiment.

(b) For the four outcomes in the sample space Ω listed in part (a), determine the associated value of each rv X, Z.

(c) Determine the support of each random variable X, Z for the experiment.

(d) Determine the probability mass function (pmf) for each random variable X, Z.

(e) Write the pmf of random variable X in closed-form by means of pattern recognition.

EX 3.2.3: Consider the following experiment: Repeatedly flip an unfair coin and observe its top face until a tail occurs.

The coin is unfair because tails occurs twice as often as heads.

Moreover, the coins flips are independent of one another.

$$\begin{aligned} \text{Let random variable} \quad X &\equiv (\#\text{ Heads Observed}) \\ \text{Let random variable} \quad Z &\equiv (\text{Is at least One Tail Observed?} \quad (1 = \text{Yes}, 0 = \text{No})) \end{aligned}$$

(a) List four possible outcomes in the sample space Ω for the experiment.

(b) For the four outcomes in the sample space Ω listed in part (a), determine the associated value of each rv X, Z.

(c) Determine the support of each random variable X, Z for the experiment.

(d) Determine the probability mass function (pmf) for each random variable X, Z.

(e) Write the pmf of random variable X in closed-form by means of pattern recognition.

EX 3.2.4: Consider the following experiment: Flip two fair coins and observe their top faces.

$$\text{Let random variable} \quad X \quad \equiv \quad (\text{\# Heads Observed})$$

Compute the following probabilities in two ways, one way using the pmf of X and the second way using the cdf of X:

$\mathbb{P}(\text{Exactly one heads}) =$

$\mathbb{P}(\text{Exactly one heads}) =$

$\mathbb{P}(\text{At most one heads}) =$

$\mathbb{P}(\text{At most one heads}) =$

$\mathbb{P}(\text{At least one heads}) =$

$\mathbb{P}(\text{At least one heads}) =$

$\mathbb{P}(\text{Either no heads or two heads}) =$

$\mathbb{P}(\text{Either no heads or two heads}) =$

EX 3.2.5: Consider the following experiment: Repeatedly flip a fair coin and observe its top face until a tail occurs. Moreover, the coins flips are independent of one another.

$$\text{Let random variable} \quad X \quad \equiv \quad (\text{\# Heads Observed})$$

Compute the following probabilities using the pmf of X:

$\mathbb{P}(\text{Exactly one heads}) =$

$\mathbb{P}(\text{At most one heads}) =$

$\mathbb{P}(\text{At least one heads}) =$

$\mathbb{P}(\text{Between two heads and six heads, inclusive}) =$

$\mathbb{P}(\text{Between two heads and six heads, exclusive}) =$

EX 3.2.6: Consider the following experiment: Repeatedly flip an unfair coin and observe its top face until a tail occurs. The coin is unfair because tails occurs twice as often as heads. Moreover, the coins flips are independent of one another.

$$\text{Let random variable} \quad X \quad \equiv \quad (\text{\# Heads Observed})$$

Compute the following probabilities using the pmf of X:

$\mathbb{P}(\text{Exactly one heads}) =$

$\mathbb{P}(\text{At most one heads}) =$

$\mathbb{P}(\text{At least one heads}) =$

$\mathbb{P}(\text{Between two heads and six heads, inclusive}) =$

$\mathbb{P}(\text{Between two heads and six heads, exclusive}) =$