EX 3.3.1: Consider the following experiment: Flip two fair coins and observe their top faces.

Let random variable $X \equiv$ (# Heads Observed)
Let random variable $Z \equiv$ (Is at least One Tail Observed? (1 = Yes, 0 = No))

Recall from the 3.2 Outline the pmf's of X and Z:

k	0	1	2
$p_X(k)$	1/4	1/2	1/4

k	0	1
$p_Z(k)$	1/4	3/4

(a) Compute the expected values of X and Z. Interpret the results.

$$\mu_X = \mathbb{E}[X] = \sum_{k \in \text{Supp}(X)} k \cdot p_X(k) = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) = (0)\left(\tfrac{1}{4}\right) + (1)\left(\tfrac{1}{2}\right) + (2)\left(\tfrac{1}{4}\right)$$

$$\mu_Z = \mathbb{E}[Z] = \sum_{k \in \text{Supp}(Z)} k \cdot p_Z(k) = 0 \cdot p_Z(0) + 1 \cdot p_Z(1) = (0)\left(\tfrac{1}{4}\right) + (1)\left(\tfrac{3}{4}\right) = \boxed{3/4}$$

$\mathbb{E}[X] = 1$ means if this experiment is performed many many times, on average expect one heads.

$\mathbb{E}[Z] = 3/4$ means if this experiment is performed many many times, on average expect at least one tails.

(since 3/4 is closer to 1 than 0, and $Z = 1$ means at least one tails is observed.)

(b) Compute the variances of X and Z.

$$\mathbb{E}[X^2] = \sum_{k \in \text{Supp}(X)} k^2 \cdot p_X(k) = 0^2 \cdot p_X(0) + 1^2 \cdot p_X(1) + 2^2 \cdot p_X(2) = (0^2)\left(\tfrac{1}{4}\right) + (1^2)\left(\tfrac{1}{2}\right) + (2^2)\left(\tfrac{1}{4}\right) = 3/2$$

$$\therefore \quad \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 3/2 - (1)^2 = \boxed{1/2}$$

$$\mathbb{E}[Z^2] = \sum_{k \in \text{Supp}(Z)} k^2 \cdot p_Z(k) = 0^2 \cdot p_Z(0) + 1^2 \cdot p_Z(1) = (0^2)\left(\tfrac{1}{4}\right) + (1^2)\left(\tfrac{3}{4}\right) = 3/4$$

$$\therefore \quad \mathbb{V}[Z] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 = 3/4 - (3/4)^2 = \boxed{3/16}$$

Observe that $\mathbb{V}[X] > \mathbb{V}[Z]$ since the histogram of X has a larger spread than the histogram of Z.

(c) Compute the standard deviations of X and Z.

$$\sigma_X = \sqrt{\mathbb{V}[X]} = \sqrt{1/2} = \boxed{1/\sqrt{2}} \approx 0.7071 \qquad \sigma_Z = \sqrt{\mathbb{V}[Z]} = \sqrt{3/16} = \boxed{\sqrt{3}/4} \approx 0.4330$$

EX 3.3.2: A music store purchased four guitars at \$150 each from a wholesaler. The store will sell each guitar for \$350.

The wholesaler refuses to buy back any unsold guitars no matter how long they sat in the store.

Let r.v. $Y \equiv$ (# guitars sold) such that Y has the pmf:

k	0	1	2	3	4
$p_Y(k)$	0.20	0.10	0.30	0.15	0.25

What is the expected **profit** for the music store? [HINT: (Profit) := (Revenue) − (Cost)]

First, compute $\mathbb{E}[Y] = \sum_{k \in \text{Supp}(Y)} k \cdot p_Y(k) = (0)(0.20) + (1)(0.10) + (2)(0.30) + (3)(0.15) + (4)(0.25) = 2.15$

$\therefore$ (Expected Profit) $= \mathbb{E}[350Y - (4)(150)] = \mathbb{E}[350Y - 600] \stackrel{(*)}{=} 350 \cdot \mathbb{E}[Y] - 600 = (350)(2.15) - 600 = \boxed{\$152.50}$

EX 3.3.3: A music store purchased four guitars at \$150 each from a wholesaler. The store will sell each guitar for \$350.

The wholesaler agrees to buy back any unsold guitars after six months for \$50 apiece.

Let r.v. $W \equiv$ (# guitars sold) such that W has the pmf:

k	0	1	2	3	4
$p_W(k)$	0.20	0.10	0.30	0.15	0.25

What is the expected **profit** for the music store? [HINT: (Profit) := (Revenue) − (Cost)]

First, compute $\mathbb{E}[W] = \sum_{k \in \text{Supp}(W)} k \cdot p_W(k) = (0)(0.20) + (1)(0.10) + (2)(0.30) + (3)(0.15) + (4)(0.25) = 2.15$

$\therefore$ (Expected Profit) $= \mathbb{E}[350W + 50(4 - W) - (4)(150)] = \mathbb{E}[350W - 400] \stackrel{(*)}{=} 350 \cdot \mathbb{E}[W] - 400 = (350)(2.15) - 400 = \boxed{\$245.00}$

$(*)$ Linearity of Expected Value: $\mathbb{E}[a \cdot X + b] = a \cdot \mathbb{E}[X] + b$