EX 4.1.1: A local gas station has three pumps, each of which can pump up to 1000 gallons per week.
Let $X \equiv$ (Total amount of gas (in 1000's of gallons) pumped at the station in a week). Then, X has the pdf:

$$f_X(x) = \begin{cases} x/2 & \text{, if } 0 \le x < 1 \\ 1 - 2kx & \text{, if } 1 \le x \le 3 \\ 0 & \text{, otherwise} \end{cases}$$

(a) What is the support of random variable X?

$\text{Supp}(X) = (\text{Set of all meaningful values of } X) = \boxed{[0,3]}$

(b) Determine the value k.

$$f_X(x) \text{ is a pdf} \implies \int_{\text{Supp}(X)} f_X(x)\,dx = 1 \implies \int_0^3 f_X(x)\,dx = 1 \implies \int_0^1 \frac{1}{2}x\,dx + \int_1^3 (1-2kx)\,dx = 1$$

$$\implies \left[\frac{1}{4}x^2\right]_{x=0}^{x=1} + \left[x - kx^2\right]_{x=1}^{x=3} = 1 \overset{FTC}{\implies} \left[\frac{1}{4}(1)^2 - \frac{1}{4}(0)^2\right] + \left[(3) - k(3)^2\right] - \left[(1) - k(1)^2\right] = 1$$

$$\implies \frac{1}{4} + 2 - 8k = 1 \implies \boxed{k = \frac{5}{32}} \qquad \therefore\ f_X(x) = \begin{cases} x/2 & \text{, if } 0 \le x < 1 \\ 1 - (5/16)x & \text{, if } 1 \le x \le 3 \\ 0 & \text{, otherwise} \end{cases}$$

(c) What is the probability that the station will pump at most 500 gallons in a week?

$$\mathbb{P}(X \le 0.5) = \int_{-\infty}^{0.5} f_X(x)\,dx = \int_0^{0.5} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_{x=0}^{x=0.5} \overset{FTC}{=} \frac{1}{4}(0.5)^2 - \frac{1}{4}(0)^2 = \boxed{0.0625}$$

(d) What is the probability that the station will pump between 200 and 600 gallons in a week?

$$\mathbb{P}(0.2 \le X \le 0.6) = \int_{0.2}^{0.6} f_X(x)\,dx = \int_{0.2}^{0.6} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_{x=0.2}^{x=0.6} \overset{FTC}{=} \frac{1}{4}(0.6)^2 - \frac{1}{4}(0.2)^2 = \boxed{0.08}$$

(e) What is the probability that the station will pump between 1000 and 2000 gallons in a week?

$$\mathbb{P}(1 \le X \le 2) = \int_1^2 f_X(x)\,dx = \int_1^2 \left(1 - \frac{5}{16}x\right)dx = \left[x - \frac{5}{32}x^2\right]_{x=1}^{x=2} \overset{FTC}{=} \left[(2) - \frac{5}{32}(2)^2\right] - \left[(1) - \frac{5}{32}(1)^2\right] = \boxed{0.84375}$$

(f) What is the probability that the station will pump between 500 and 3000 gallons in a week?

$$\mathbb{P}(0.5 \le X \le 3) = \int_{0.5}^3 f_X(x)\,dx = \int_{0.5}^1 \frac{1}{2}x\,dx + \int_1^3 \left(1 - \frac{5}{16}x\right)dx = \left[\frac{1}{4}x^2\right]_{x=0.5}^{x=1} + \left[x - \frac{5}{32}x^2\right]_{x=1}^{x=3}$$

$$\overset{FTC}{=} \left[\frac{1}{4}(1)^2 - \frac{1}{4}(0.5)^2\right] + \left[(3) - \frac{5}{32}(3)^2\right] - \left[(1) - \frac{5}{32}(1)^2\right] = \boxed{0.9375}$$

(g) What is the probability that the station will pump either at most 500 gallons or at least 2000 gallons in a week?

$$\mathbb{P}(X \le 0.5 \text{ or } X \ge 2) = \mathbb{P}(X \le 0.5 \cup X \ge 2) \overset{PD}{=} \mathbb{P}(X \le 0.5) + \mathbb{P}(X \ge 2) = \int_{-\infty}^{0.5} f_X(x)\,dx + \int_2^{\infty} f_X(x)\,dx$$

$$= \int_0^{0.5} \frac{1}{2}x\,dx + \int_2^3 \left(1 - \frac{5}{16}x\right)dx = \left[\frac{1}{4}x^2\right]_{x=0}^{x=0.5} + \left[x - \frac{5}{32}x^2\right]_2^3$$

$$\overset{FTC}{=} \left[\frac{1}{4}(0.5)^2 - \frac{1}{4}(0)^2\right] + \left[(3) - \frac{5}{32}(3)^2\right] - \left[(2) - \frac{5}{32}(2)^2\right] = \boxed{0.28125}$$

EX 4.1.2: Let $X \equiv$ (Lifetime (in hundreds of hours) of a particular brand of amplifier). Then, the pdf of X is:

$$f_X(x) = \begin{cases} 4/x^2 & \text{, if } x > 4 \\ 0 & \text{, if } x \leq 4 \end{cases}$$

(a) What is the support of random variable X?

Supp(X) = (Set of all meaningful values of X) = $\boxed{[0, \infty)}$ *OR* Supp(X) = (Set of all values of X s.t. $f_X > 0$) = $\boxed{(4, \infty)}$

(b) What is the probability that such an amplifier functions at most 300 hours?

$$\mathbb{P}(X \leq 3) = \int_{-\infty}^{3} f_X(x)\ dx = \int_{-\infty}^{3} (0)\ dx = \boxed{0}$$

(c) What is the probability that such an amplifier functions between 300 and 1000 hours?

$$\mathbb{P}(3 \leq X \leq 10) = \int_{3}^{10} f_X(x)\ dx = \int_{4}^{10} \frac{4}{x^2}\ dx = \left[-\frac{4}{x}\right]_{x=4}^{x=10} \overset{FTC}{=} -\frac{4}{(10)} - \left(-\frac{4}{(4)}\right) = \frac{3}{5} = \boxed{0.60}$$

(d) What is the probability that such an amplifier functions at least 1000 hours?

$$\mathbb{P}(X \geq 10) = \int_{10}^{\infty} f_X(x)\ dx = \int_{10}^{\infty} \frac{4}{x^2}\ dx = \left[-\frac{4}{x}\right]_{x=10}^{x\to\infty} \overset{FTC}{=} \left(\lim_{x\to\infty} -\frac{4}{x}\right) - \left(-\frac{4}{(10)}\right) = 0 + \frac{2}{5} = \frac{2}{5} = \boxed{0.40}$$

(e) What is the probability that at least two of three such amplifiers function at least 1000 hours?

(Assume the lifetimes of the three amplifiers are **independent** of one another.)

Let $X_1, X_2, X_3 \sim$ pdf f_X. Then:

$$\mathbb{P}\left[(X_1 < 10 \text{ and } X_2, X_3 \geq 10) \text{ or } (X_2 < 10 \text{ and } X_1, X_3 \geq 10) \text{ or } (X_3 < 10 \text{ and } X_1, X_2 \geq 10) \text{ or } (X_1, X_2, X_3 \geq 10)\right]$$
$$= \mathbb{P}\left[(X_1 < 10 \cap X_2 \geq 10 \cap X_3 \geq 10) \cup (X_2 < 10 \cap X_1 \geq 10 \cap X_3 \geq 10) \cup (X_3 < 10 \cap X_1 \geq 10 \cap X_2 \geq 10) \cup (X_1 \geq 10 \cap X_2 \geq 10 \cap X_3 \geq 10)\right]$$
$$\overset{PD}{=} \mathbb{P}(X_1 < 10 \cap X_2 \geq 10 \cap X_3 \geq 10) + \mathbb{P}(X_2 < 10 \cap X_1 \geq 10 \cap X_3 \geq 10) + \mathbb{P}(X_3 < 10 \cap X_1 \geq 10 \cap X_2 \geq 10) + \mathbb{P}(X_1 \geq 10 \cap X_2 \geq 10 \cap X_3 \geq 10)$$
$$\overset{IND}{=} \mathbb{P}(X_1 < 10)\mathbb{P}(X_2 \geq 10)\mathbb{P}(X_3 \geq 10) + \mathbb{P}(X_2 < 10)\mathbb{P}(X_1 \geq 10)\mathbb{P}(X_3 \geq 10) + \mathbb{P}(X_3 < 10)\mathbb{P}(X_1 \geq 10)\mathbb{P}(X_2 \geq 10) + \mathbb{P}(X_1 \geq 10)\mathbb{P}(X_2 \geq 10)\mathbb{P}(X_3 \geq 10)$$
$$= (0.60)(0.40)(0.40) + (0.60)(0.40)(0.40) + (0.60)(0.40)(0.40) + (0.40)(0.40)(0.40) = \boxed{0.352}$$

(f) What is the probability that at exactly two of three such amplifiers function at least 1000 hours?

(Assume the lifetimes of the three amplifiers are **independent** of one another.)

Let $X_1, X_2, X_3 \sim$ pdf f_X. Then:

$$\mathbb{P}\left[(X_1 < 10 \text{ and } X_2, X_3 \geq 10) \text{ or } (X_2 < 10 \text{ and } X_1, X_3 \geq 10) \text{ or } (X_3 < 10 \text{ and } X_1, X_2 \geq 10)\right]$$
$$= \mathbb{P}\left[(X_1 < 10 \cap X_2 \geq 10 \cap X_3 \geq 10) \cup (X_2 < 10 \cap X_1 \geq 10 \cap X_3 \geq 10) \cup (X_3 < 10 \cap X_1 \geq 10 \cap X_2 \geq 10)\right]$$
$$\overset{PD}{=} \mathbb{P}(X_1 < 10 \cap X_2 \geq 10 \cap X_3 \geq 10) + \mathbb{P}(X_2 < 10 \cap X_1 \geq 10 \cap X_3 \geq 10) + \mathbb{P}(X_3 < 10 \cap X_1 \geq 10 \cap X_2 \geq 10)$$
$$\overset{IND}{=} \mathbb{P}(X_1 < 10)\mathbb{P}(X_2 \geq 10)\mathbb{P}(X_3 \geq 10) + \mathbb{P}(X_2 < 10)\mathbb{P}(X_1 \geq 10)\mathbb{P}(X_3 \geq 10) + \mathbb{P}(X_3 < 10)\mathbb{P}(X_1 \geq 10)\mathbb{P}(X_2 \geq 10)$$
$$= (0.60)(0.40)(0.40) + (0.60)(0.40)(0.40) + (0.60)(0.40)(0.40) = \boxed{0.288}$$

EX 4.1.3: A system consisting of one component and a spare component can function for a random amount of time.

Let $X \equiv$ (Length of time (in months) that the system functions) Then, X has the following density:

$$f_X(x) = \begin{cases} \frac{x}{16}e^{-x/4} & \text{, if } x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

What is the probability that the system functions for at least 8 months?

$$\mathbb{P}(X \geq 8) = \int_8^\infty f_X(x)\ dx = \int_8^\infty \left(\frac{x}{16}\right) e^{-x/4}\ dx \overset{IBP}{=} \left[\left(\frac{x}{16}\right)\left(-4e^{-x/4}\right)\right]_{x=8}^{x\to\infty} - \int_8^\infty \left(-4e^{-x/4}\right)\left(\frac{1}{16}\ dx\right)$$

$$= \left[-\frac{x}{4}e^{-x/4}\right]_{x=8}^{x\to\infty} + \frac{1}{4}\int_8^\infty e^{-x/4}\ dx = \left[-\frac{x}{4}e^{-x/4}\right]_{x=8}^{x\to\infty} + \frac{1}{4}\Big[-4e^{-x/4}\Big]_{x=8}^{x\to\infty}$$

$$\overset{FTC}{=} \left[\left(\lim_{x\to\infty} -\frac{x}{4}e^{-x/4}\right) - \left(-\frac{(8)}{4}e^{-(8)/4}\right)\right] + \frac{1}{4}\left[\left(\lim_{x\to\infty} -4e^{-x/4}\right) - \left(-4e^{-(8)/4}\right)\right]$$

$$\overset{(*)}{=} \left[0 - \left(-2e^{-2}\right)\right] + \frac{1}{4}\left[0 - \left(-4e^{-2}\right)\right] = 2e^{-2} + e^{-2} = 3e^{-2} \approx \boxed{0.40601}$$

Integration by Parts (IBP): Let $\begin{cases} u = \frac{x}{16} \\ dv = e^{-x/4}\ dx \end{cases} \implies \begin{cases} du = \frac{d}{dx}\left[\frac{x}{16}\right]\ dx \\ v = \int e^{-x/4}\ dx \end{cases} \implies \begin{cases} du = \frac{1}{16}\ dx \\ v = -4e^{-x/4} \end{cases}$

$$\therefore \int_8^\infty \left(\frac{x}{16}\right) e^{-x/4}\ dx \overset{IBP}{=} \Big[uv\Big]_{x=8}^{x\to\infty} - \int_8^\infty v\ du = \left[\left(\frac{x}{16}\right)\left(-4e^{-x/4}\right)\right]_{x=8}^{x\to\infty} - \int_8^\infty \left(-4e^{-x/4}\right)\left(\frac{1}{16}\ dx\right)$$

(∗) L'Hospital's Rule (LHOP): $\lim_{x\to\infty} -\frac{x}{4}e^{-x/4} = \lim_{x\to\infty} \frac{-x}{4e^{x/4}} \overset{NS}{=} -\frac{\infty}{\infty} \overset{LHOP}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[-x]}{\frac{d}{dx}[4e^{x/4}]} = \lim_{x\to\infty} \frac{-1}{e^{x/4}} = 0$

Note that the integral $\int_8^\infty e^{-x/4}\ dx$ can be computed using a **change of variables**:

CV: Let $u = -\frac{x}{4}$. Then, $du = \frac{d}{dx}\left[-\frac{x}{4}\right] \implies du = -\frac{1}{4}\ dx \implies dx = -4\ du$

Moreover, $u(\infty) = \lim_{x\to\infty}\left(-\frac{x}{4}\right) \overset{NS}{=} -\frac{(\infty)}{4} = -\infty$ and $u(8) = -\frac{(8)}{4} = -2$

$$\therefore \int_8^\infty e^{-x/4}\ dx \overset{CV}{=} \int_{-2}^{-\infty} e^u\ (-4\ du) = -4\int_{-2}^{-\infty} e^u\ du = -4\Big[e^u\Big]_{u=-2}^{u\to-\infty} \overset{FTC}{=} -4\left[\lim_{u\to-\infty} e^u - e^{(-2)}\right] = -4\left[0 - e^{-2}\right] = 4e^{-2}$$