CONTINUOUS RANDOM VARIABLES, PDF'S [DEVORE 4.1]

- **CONTINUOUS RANDOM VARIABLES:** X is a **continuous random variable** $\iff$ Supp(X) is uncountable.

  i.e. The meaningful values of X comprise an interval or union of intervals or the entire real line $\mathbb{R}$.

- **CONTINUOUS RANDOM VARIABLES & THEIR SUPPORTS (EXAMPLES):**

  – Experiment: Randomly choose a point on unit square centered at $(0, 0)$.

$$\begin{array}{cccccc} X & \equiv & x\text{-coordinate} & \Longrightarrow & \text{Supp}(X) & = & \left[-\frac{1}{2}, \frac{1}{2}\right] \\ Y & \equiv & \text{Magnitude of } y\text{-coord} & \Longrightarrow & \text{Supp}(Y) & = & \left[0, \frac{1}{2}\right] \\ Z & \equiv & \text{Distance to } (0,0) & \Longrightarrow & \text{Supp}(Z) & = & [0, 1/\sqrt{2}] \\ W & \equiv & \text{Distance to farthest corner} & \Longrightarrow & \text{Supp}(W) & = & [1/2, \sqrt{2}] \end{array}$$

  – Experiment: Randomly select two adults in a busy airport.

$$\begin{array}{ccccccc} X & \equiv & \text{Height of } 1^{st} \text{ adult (in ft)} & \Longrightarrow & \text{Supp}(X) & = & [1.75, 9.00] \\ Y & \equiv & \text{Height of } 1^{st} \text{ adult (in cm)} & \Longrightarrow & \text{Supp}(Y) & = & [54, 275] \\ Z & \equiv & \text{Height of } 1^{st} \text{ adult (in m)} & \Longrightarrow & \text{Supp}(Z) & = & [0.54, 2.75] \\ W & \equiv & \begin{array}{c}\text{Height difference of} \\ \text{the two adults (in ft)}\end{array} & \Longrightarrow & \text{Supp}(W) & = & [-7.25, 7.25] \end{array}$$

  – Experiment: Take note of the time when a call center phone rings.

$$\begin{array}{ccccccc} X & \equiv & \begin{array}{c}\text{Elapsed time until} \\ \text{next phone call (in secs)}\end{array} & \Longrightarrow & \text{Supp}(X) & = & (0, \infty) \\ Y & \equiv & \begin{array}{c}\text{Elapsed time until} \\ \text{next phone call (in hours)}\end{array} & \Longrightarrow & \text{Supp}(Y) & = & (0, \infty) \end{array}$$

- **MEASUREMENTS ARE NEVER 100% ACCURATE:**

  A height is reported as, say, 251.6 cm instead of 251.5786800233791427 cm!

  Also, there are a finite # of people, hence a finite # of heights to measure!

  So why is a r.v. for a person's height considered continuous & not discrete????

  Well, it turns out that in practice:

  – Very large populations are well-approximated by continuous distributions.

  – It's easier to mathematically work with continuous r.v.'s than discrete r.v.'s:

$$\text{e.g.} \quad \text{It's far easier to compute} \quad \int_1^\infty \frac{1}{x^2}\, dx \quad \text{than} \quad \sum_{k=1}^\infty \frac{1}{k^2}$$

- **PROBABILITY DENSITY FUNCTION (PDF) OF A CONTINUOUS R.V.:** Let X be a **continuous** rv.

  Then, its **probability density function (pdf)**, denoted $f_X(x)$, is a function of the possible values of X s.t.:

$$\begin{array}{lll} \text{Non-negativity on its Support:} & f_X(x) \geq 0 & \forall x \in \text{Supp}(X) \\ \text{Zero outside of its Support:} & f_X(x) = 0 & \forall x \notin \text{Supp}(X) \\ \text{Universal Integral of Unity:} & \displaystyle\int_{\text{Supp}(X)} f_X(x)\, dx = 1 & \end{array}$$

  The graph of $f_X(x)$ is often called the **density curve** for random variable X.

- **PROPERTIES OF A CONTINUOUS R.V.:** Let X be a **continuous** r.v. with pdf $f_X(x)$. Let $a < b$. Then:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\mathbb{P}(X = a) = 0, \quad \mathbb{P}(X > b) = \mathbb{P}(X \geq b), \quad \mathbb{P}(X < a) = \mathbb{P}(X \leq a)$$

$$\mathbb{P}(a < X < b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a \leq X \leq b)$$

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EX 4.1.1: A local gas station has three pumps, each of which can pump up to 1000 gallons per week.

Let $X \equiv$ (Total amount of gas (in 1000's of gallons) pumped at the station in a week). Then, X has the pdf:

$$f_X(x) = \begin{cases} x/2 & \text{, if } 0 \le x < 1 \\ 1 - 2kx & \text{, if } 1 \le x \le 3 \\ 0 & \text{, otherwise} \end{cases}$$

(a) What is the support of random variable X?

(b) Determine the value k.

(c) What is the probability that the station will pump at most 500 gallons in a week?

(d) What is the probability that the station will pump between 200 and 600 gallons in a week?

(e) What is the probability that the station will pump between 1000 and 2000 gallons in a week?

(f) What is the probability that the station will pump between 500 and 3000 gallons in a week?

(g) What is the probability that the station will pump either at most 500 gallons or at least 2000 gallons in a week?

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EX 4.1.2: Let $X \equiv$ (Lifetime (in hundreds of hours) of a particular brand of amplifier). Then, the pdf of X is:

$$f_X(x) = \begin{cases} 4/x^2 & , \text{ if } x > 4 \\ 0 & , \text{ if } x \leq 4 \end{cases}$$

(a) What is the support of random variable X?

(b) What is the probability that such an amplifier functions at most 300 hours?

(c) What is the probability that such an amplifier functions between 300 and 1000 hours?

(d) What is the probability that such an amplifier functions at least 1000 hours?

(e) What is the probability that at least two of three such amplifiers function at least 1000 hours?
(Assume the lifetimes of the three amplifiers are **independent** of one another.)

(f) What is the probability that at exactly two of three such amplifiers function at least 1000 hours?
(Assume the lifetimes of the three amplifiers are **independent** of one another.)

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EX 4.1.3: A system consisting of one component and a spare component can function for a random amount of time.

Let $X \equiv$ (Length of time (in months) that the system functions)   Then, X has the following density:

$$f_X(x) = \begin{cases} \frac{x}{16}e^{-x/4} & \text{, if } x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

What is the probability that the system functions for at least 8 months?

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