EX 4.2.1: A local gas station has three pumps, each of which can pump up to 1000 gallons per week.

Let $X \equiv$ (Total amount of gas (in 1000's of gallons) pumped at the station in a week). Then, X has the pdf:

$$f_X(x) = \begin{cases} x/2 & \text{, if } 0 \leq x < 1 \\ 1-(5/16)x & \text{, if } 1 \leq x \leq 3 \\ 0 & \text{, otherwise} \end{cases}$$

(a) What is the support of random variable X?

Supp(X) = (Set of all meaningful values of X) = $\boxed{[0,3]}$

(b) Determine the cdf of X, $F_X(x)$.

$$F_X(x) = \begin{cases} 0 & \text{, if } x<0 \\ \int_{-\infty}^{x} f_X(t)\,dt & \text{, if } 0 \leq x < 1 \\ \int_{-\infty}^{x} f_X(t)\,dt & \text{, if } 1 \leq x \leq 3 \\ 1 & \text{, if } 3 < x \end{cases} = \begin{cases} 0 & \text{, if } x<0 \\ \int_0^x \frac{t}{2}\,dt & \text{, if } 0 \leq x \leq 1 \\ \int_0^1 \frac{t}{2}\,dt + \int_0^x (1-\frac{5}{16}t)\,dt & \text{, if } 1 \leq x \leq 3 \\ 1 & \text{, if } 3 < x \end{cases} \implies F_X(x) = \boxed{\begin{cases} 0 & \text{, if } x<0 \\ \frac{1}{4}x^2 & \text{, if } 0 \leq x \leq 1 \\ x-\frac{5}{32}x^2-\frac{19}{32} & \text{, if } 1 \leq x \leq 3 \\ 1 & \text{, if } 3 < x \end{cases}}$$

For $0 \leq x < 1$: $\displaystyle\int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{1}{2}t\,dt = \left[\frac{1}{4}t^2\right]_{t=0}^{t=x} \overset{FTC}{=} \frac{1}{4}(x)^2 - \frac{1}{4}(0)^2 = \frac{1}{4}x^2$

For $1 \leq x \leq 3$: $\displaystyle\int_{-\infty}^{x} f_X(t)\,dt = \int_0^1 \frac{1}{2}t\,dt + \int_1^x \left(1-\frac{5}{16}t\right)\,dt = \left[\frac{1}{4}t^2\right]_{t=0}^{t=1} + \left[t-\frac{5}{32}t^2\right]_{t=1}^{t=x} \overset{FTC}{=} x - \frac{5}{32}x^2 - \frac{19}{32}$

(c) **Using the cdf**, what is the probability that the station will pump at most 500 gallons in a week?

$\mathbb{P}(X \leq 0.5) = F_X(0.5) = \frac{1}{4}(0.5)^2 = \boxed{\mathbf{0.0625}}$

(d) **Using the cdf**, what is the probability that the station will pump between 1000 and 2000 gallons in a week?

$\mathbb{P}(1 \leq X \leq 2) = \mathbb{P}(X \leq 2) - \mathbb{P}(X \leq 1) = F_X(2) - F_X(1) = \left[(2) - \frac{5}{32}(2)^2\right] - \left[(1) - \frac{5}{32}(1)^2\right] = \frac{17}{32} = \boxed{\mathbf{0.53125}}$

(e) What is the expected amount of gas pumped by the station in a week?

$$\begin{aligned}\mathbb{E}[X] &= \int_{\text{Supp}(X)} x \cdot f_X(x)\,dx = \int_0^1 x \cdot \left(\frac{x}{2}\right)\,dx + \int_1^3 x \cdot \left(1-\frac{5}{16}x\right)\,dx = \int_0^1 \frac{1}{2}x^2\,dx + \int_1^3 \left(x - \frac{5}{16}x^2\right)\,dx \\ &= \left[\frac{1}{6}x^3\right]_{x=0}^{x=1} + \left[\frac{1}{2}x^2 - \frac{5}{48}x^3\right]_{x=1}^{x=3} \overset{FTC}{=} \left[\frac{1}{6}(1)^3 - \frac{1}{6}(0)^3\right] + \left[\left(\frac{1}{2}(3)^2 - \frac{5}{48}(3)^3\right) - \left(\frac{1}{2}(1)^2 - \frac{5}{48}(1)^3\right)\right] = \frac{35}{24} \approx 1.45833\end{aligned}$$

$\therefore$ The expected amount of gas pumped by the station in a week is about $(1.45833)(1000) =$ **1458.33 gallons**

(f) What is the variance of the amount of gas pumped by the station in a week?

$$\begin{aligned}\mathbb{E}[X^2] &= \int_{\text{Supp}(X)} x^2 \cdot f_X(x)\,dx = \int_0^1 x^2 \cdot \left(\frac{x}{2}\right)\,dx + \int_1^3 x^2 \cdot \left(1-\frac{5}{16}x\right)\,dx = \int_0^1 \frac{1}{2}x^3\,dx + \int_1^3 \left(x^2 - \frac{5}{16}x^3\right)\,dx \\ &= \left[\frac{1}{8}x^4\right]_{x=0}^{x=1} + \left[\frac{1}{3}x^3 - \frac{5}{64}x^4\right]_{x=1}^{x=3} \overset{FTC}{=} \left[\frac{1}{8}(1)^4 - \frac{1}{8}(0)^4\right] + \left[\left(\frac{1}{3}(3)^3 - \frac{5}{64}(3)^4\right) - \left(\frac{1}{3}(1)^3 - \frac{5}{64}(1)^4\right)\right] = \frac{61}{24} \approx 2.54167\end{aligned}$$

$$\implies \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{61}{24} - \left(\frac{35}{24}\right)^2 = \frac{239}{576} \approx 0.414930556$$

$\therefore$ The variance of amount of gas pumped by station in a week is about $(0.414930556)(1000^2) =$ **414930.556 gallons^2**

(g) What is the standard deviation of the amount of gas pumped by the station in a week?

$\sigma_X = \sqrt{\mathbb{V}[X]} = \sqrt{0.414930556} \approx 0.64415$

$\therefore$ The std dev of amount of gas pumped by the station in a week is about $(0.64415)(1000) =$ **644.15 gallons**

EX 4.2.2: Let $X \equiv$ (Monthly salary ($) where few workers earn very high monthly salaries). Then, the pdf of X is:

$$f_X(x) = \begin{cases} (3 \times 10^9)/x^4 & \text{, if } x > 1000 \\ 0 & \text{, if } x \leq 1000 \end{cases}$$

(a) What is the support of random variable X?

Supp(X) = (Set of all meaningful values of X) = $\boxed{[0, \infty)}$ *OR* Supp(X) = (Set of all values of X s.t. $f_X > 0$) = $\boxed{(1000, \infty)}$

(b) Determine the cdf of X, $F_X(x)$.

$$F_X(x) = \begin{cases} 0 & \text{, if } x \leq 1000 \\ \int_{-\infty}^{x} f_X(t)\ dt & \text{, if } x > 1000 \end{cases} = \begin{cases} 0 & \text{, if } x \leq 1000 \\ \int_{1000}^{x} f_X(t)\ dt & \text{, if } x > 1000 \end{cases} \implies F_X(x) = \boxed{\begin{cases} 0 & \text{, if } x \leq 1000 \\ 1 - \dfrac{10^9}{x^3} & \text{, if } x > 1000 \end{cases}}$$

$$\int_{-\infty}^{x} f_X(t)\ dt = \int_{1000}^{x} \frac{3 \times 10^9}{t^4}\ dt = \left[\frac{3 \times 10^9}{(-3)t^3}\right]_{t=1000}^{t=x} = \left[-\frac{1 \times 10^9}{t^3}\right]_{t=1000}^{t=x} \stackrel{FTC}{=} \left(-\frac{10^9}{x^3}\right) - \left(-\frac{10^9}{1000^3}\right) = \frac{10^9}{10^9} - \frac{10^9}{x^3}$$

(c) **Using the cdf**, what is the probability that a worker earns between \$2000 and \$3000 per month?

$$\begin{aligned} \mathbb{P}(2000 \leq X \leq 3000) &= \mathbb{P}(X \leq 3000) - \mathbb{P}(X \leq 2000) = F_X(3000) - F_X(2000) = \left(1 - \frac{10^9}{(3000)^3}\right) - \left(1 - \frac{10^9}{(2000)^3}\right) \\ &= \frac{10^9}{(2 \times 10^3)^3} - \frac{10^9}{(3 \times 10^3)^3} = \frac{1 \times 10^9}{8 \times 10^9} - \frac{1 \times 10^9}{27 \times 10^9} = \frac{1}{8} - \frac{1}{27} = \frac{19}{216} \approx \boxed{\mathbf{0.08796}} \end{aligned}$$

(d) What is the mean monthly salary, μ_X?

$$\begin{aligned} \mu_X &= \mathbb{E}[X] = \int_{\text{Supp}(X)} x \cdot f_X(x)\ dx = \int_{1000}^{\infty} x \cdot \frac{3 \times 10^9}{x^4}\ dx = \int_{1000}^{\infty} \frac{3 \times 10^9}{x^3}\ dx = \left[-\frac{3 \times 10^9}{2x^2}\right]_{x=1000}^{x\to\infty} = \left[-\frac{1.5 \times 10^9}{x^2}\right]_{x=1000}^{x\to\infty} \\ &\stackrel{FTC}{=} \left[\lim_{x\to\infty}\left(-\frac{1.5 \times 10^9}{x^2}\right)\right] - \left(-\frac{1.5 \times 10^9}{(1000)^2}\right) = 0 + \frac{1.5 \times 10^9}{(1 \times 10^3)^2} = \frac{1.5 \times 10^9}{1 \times 10^6} = 1.5 \times 10^3 = \boxed{\mathbf{\$1500.00}} \end{aligned}$$

(e) What is the variance of the monthly salary, σ_X^2?

$$\begin{aligned} \mathbb{E}[X^2] &= \int_{\text{Supp}(X)} x^2 \cdot f_X(x)\ dx = \int_{1000}^{\infty} x^2 \cdot \frac{3 \times 10^9}{x^4}\ dx = \int_{1000}^{\infty} \frac{3 \times 10^9}{x^2}\ dx = \left[-\frac{3 \times 10^9}{x}\right]_{x=1000}^{x\to\infty} \\ &\stackrel{FTC}{=} \left[\lim_{x\to\infty}\left(-\frac{3 \times 10^9}{x}\right)\right] - \left(-\frac{3 \times 10^9}{(1000)}\right) = 0 + \frac{3 \times 10^9}{1 \times 10^3} = 3 \times 10^6 = 3000000 \end{aligned}$$

$$\therefore\ \sigma_X^2 = \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 3000000 - (1500)^2 = \boxed{\mathbf{750{,}000\ dollars^2}}$$

(f) What is the median monthly salary, $\widetilde{\mu}_X$?

Solve the following equation: $0.5 = F(\widetilde{\mu}_X) \implies 1 - \dfrac{10^9}{(\widetilde{\mu}_X)^3} = 0.5 \implies \dfrac{10^9}{(\widetilde{\mu}_X)^3} = 0.5 \implies \widetilde{\mu}_X = \sqrt[3]{\dfrac{10^9}{0.5}} \approx \boxed{\mathbf{\$1259.92}}$

(g) What is the probability that a worker's monthly salary is within 2 standard deviations of the mean value?

$$\begin{aligned} \mathbb{P}(|X - \mu_X| \leq 2\sigma_X) &= \mathbb{P}(-2\sigma_X \leq X - \mu_X \leq 2\sigma_X) \\ &= \mathbb{P}(\mu_X - 2\sigma_X \leq X \leq \mu_X + 2\sigma_X) = \mathbb{P}(X \leq \mu_X + 2\sigma_X) - \mathbb{P}(X \leq \mu_X - 2\sigma_X) \\ &= F_X(\mu_X + 2\sigma_X) - F_X(\mu_X - 2\sigma_X) = F_X\left(1500 + 2(866.0254)\right) - F_X\left(1500 - 2(866.0254)\right) \\ &= F_X(3232.0508) - F_X(-232.0508) = \left(1 - \frac{10^9}{(3232.0508)^3}\right) - 0 \approx \boxed{\mathbf{0.97038}} \end{aligned}$$

EX 4.2.3: Given the following cdf of X: $F_X(x) = \begin{cases} 0 & \text{, if } x \le 0 \\ (x/4)[1+\ln(4/x)] & \text{, if } 0 < x \le 4 \\ 1 & \text{, if } x > 4 \end{cases}$

What is the corresponding pdf of X, $f_X(x)$?

$$f_X(x) = F'_X(x) = \frac{d}{dx}\left[\left(\frac{x}{4}\right)\left(1+\ln\left(\frac{4}{x}\right)\right)\right] \overset{(*)}{=} \frac{1}{4}\left[1+\ln\left(\frac{4}{x}\right)\right] + \left(\frac{x}{4}\right)\cdot\frac{d}{dx}\left[1+\ln\left(\frac{4}{x}\right)\right]$$

$$= \frac{1}{4}\left[1+\ln\left(\frac{4}{x}\right)\right] + \left(\frac{x}{4}\right)\left[0+\left(\frac{x}{4}\right)\cdot\frac{d}{dx}\left[\frac{4}{x}\right]\right] = \frac{1}{4}\left[1+\ln\left(\frac{4}{x}\right)\right] + \frac{x^2}{16}\left[-\frac{4}{x^2}\right] = \frac{1}{4}+\frac{1}{4}\ln\left(\frac{4}{x}\right) - \frac{1}{4} = \frac{1}{4}\ln\left(\frac{4}{x}\right)$$

$$\therefore \boxed{f_X(x) = \begin{cases} \frac{1}{4}\ln\left(\frac{4}{x}\right) & \text{, if } 0 < x \le 4 \\ 0 & \text{, otherwise} \end{cases}} \quad \text{OR} \quad (**) \boxed{f_X(x) = \begin{cases} \frac{1}{4}[\ln 4 - \ln x] & \text{, if } 0 < x \le 4 \\ 0 & \text{, otherwise} \end{cases}}$$

$(*)$ Product Rule (from Calculus I): $\frac{d}{dx}\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x)$

$(**)$ Logarithm Properties: $\ln(AB) = \ln A + \ln B$ $\quad \ln(A/B) = \ln A - \ln B$ $\quad \ln(A^k) = k\ln A$

EX 4.2.4: Given the following cdf of X: $F_X(x) = e^{-e^{-x}}$ for $-\infty < x < \infty$

(a) Show that $\lim_{x\to-\infty} F_X(x) = 0$.

$$\lim_{x\to-\infty} F_X(x) = \lim_{x\to-\infty} e^{-e^{-x}} \overset{NS}{=} e^{-e^{-(-\infty)}} = e^{-e^{\infty}} = e^{-\infty} = 0 \checkmark$$

(b) Show that $\lim_{x\to\infty} F_X(x) = 1$.

$$\lim_{x\to\infty} F_X(x) = \lim_{x\to\infty} e^{-e^{-x}} \overset{NS}{=} e^{-e^{-(\infty)}} = e^{-e^{-\infty}} = e^{-(0)} = e^0 = 1 \checkmark$$

(c) What is the corresponding pdf of X, $f_X(x)$?

$$f_X(x) = F'_X(x) = \frac{d}{dx}\left[e^{-e^{-x}}\right] \overset{(*)}{=} \left(e^{-e^{-x}}\right)\cdot\frac{d}{dx}\left[-e^{-x}\right] \overset{(*)}{=} e^{-x}\cdot e^{-e^{-x}}$$

$$\therefore \boxed{f_X(x) = e^{-x}\cdot e^{-e^{-x}} \text{ for } -\infty < x < \infty}$$

$(*)$ Chain Rule (from Calculus I): $\frac{d}{dx}\Big[f[g(x)]\Big] = f'[g(x)]\cdot g'(x)$

(d) What is the median of X, $\widetilde{\mu}_X$?

Solve the equation: $0.5 = F(\widetilde{\mu}_X)$

$$\implies e^{-e^{(\widetilde{\mu}_X)}} = 0.5 \implies -e^{-(\widetilde{\mu}_X)} = \ln(0.5)$$
$$\implies e^{-\widetilde{\mu}_X} = -\ln(0.5) = -\ln(1/2) \overset{(*)}{=} \ln((1/2)^{-1}) = \ln 2$$
$$\implies e^{-\widetilde{\mu}_X} = \ln 2 \implies -\widetilde{\mu}_X = \ln(\ln 2)$$
$$\implies \widetilde{\mu}_X = -\ln(\ln 2) \approx \boxed{\mathbf{0.36651}}$$

$(*)$ Logarithm Properties: $\ln(AB) = \ln A + \ln B$ $\quad \ln(A/B) = \ln A - \ln B$ $\quad \ln(A^k) = k\ln A$

(e) What is the 80th percentile of X, $x_{0.80}$?

Solve the equation: $0.80 = F(x_{0.80})$

$$\implies e^{-e^{(x_{0.80})}} = 0.8 \implies -e^{-(x_{0.80})} = \ln(0.8)$$
$$\implies e^{-x_{0.80}} = -\ln(0.8) = -\ln(4/5) \overset{(*)}{=} \ln((4/5)^{-1}) = \ln(5/4)$$
$$\implies e^{-x_{0.80}} = \ln(5/4) \implies -x_{0.80} = \ln(\ln(5/4))$$
$$\implies x_{0.80} = -\ln(\ln(5/4)) \approx \boxed{\mathbf{1.49994}}$$

$(*)$ Logarithm Properties: $\ln(AB) = \ln A + \ln B$ $\quad \ln(A/B) = \ln A - \ln B$ $\quad \ln(A^k) = k\ln A$