CONTINUOUS R.V.'S: CDF'S, EXPECTATION, VARIANCE [DEVORE 4.2]

- **CDF OF A CONTINOUS R.V.:** Let X be a **continuous** random variable.

  Then, its **cdf**, denoted as $F_X(x)$, is defined as $F_X(x) := \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(\xi)\ d\xi = \int_{-\infty}^{x} f_X(t)\ dt$

  Note that ξ is the lower-case Greek letter xi, pronounced "(kuh)-SEE".

  Moreover, if you find that writing the Greek letter ξ looks like chaotic scribbles, use t instead.

- **CDF AXIOMS:** Let X be a **continuous** random variable. Then, its **cdf** $F_X(x)$, satisfies

  Eventually Zero (One) to the Left (Right): $\lim_{x \to -\infty} F_X(x) = 0, \qquad \lim_{x \to +\infty} F_X(x) = 1$

  Non-decreasing: $x_1 \leq x_2 \implies F_X(x_1) \leq F_X(x_2)$

  Continuous: $F_X \in C(\mathbb{R})$

- **OBTAINING PDF FROM CDF:** Let X be a **continuous** r.v. with pdf $f_X(x)$ and cdf $F_X(x)$. Then $F_X'(x) = f_X(x)$

- **EXPECTED VALUE (MEAN) OF A CONTINUOUS R.V.:** Let X be a **continuous** r.v. with pdf $f_X(x)$.

  Then the **expected value** (AKA **mean**) of X is $\mathbb{E}[X] := \int_{\mathrm{Supp}(X)} x \cdot f_X(x)\ dx$

  It's possible (but rare) that the expected value is **infinite**: $\mathbb{E}[X] = \infty$

  NOTATION: The expected value of X is alternatively denoted by $\overline{X}$ or μ_X.

- **EXPECTED VALUE OF A FUNCTION OF CONTINUOUS R.V.:** Let X be a continuous rv with pdf $f_X(x)$.

  Let $h(x)$ be a single-variable function. Then the **expected value** (AKA **mean**) of $h(X)$ is

$$\mathbb{E}[h(X)] := \int_{\mathrm{Supp}(X)} h(x) \cdot f_X(x)\ dx$$

  It's possible (but rare) that the expected value is **infinite**: $\mathbb{E}[h(X)] = \pm\infty$

  NOTATION: The expected value of $h(X)$ is alternatively denoted by $\mu_{h(X)}$.

- **VARIANCE OF A CONTINUOUS R.V.:** Let X be a continuous rv with pdf $f_X(x)$ and mean μ_X.

  Then the **variance** of X is

$$\mathbb{V}[X] := \mathbb{E}[(X - \mu_X)^2] = \int_{\mathrm{Supp}(X)} (x - \mu_X)^2 \cdot f_X(x)\ dx$$

  It's possible (but rare) that the variance is **infinite**: $\mathbb{V}[X] = \infty$

  NOTATION: The variance of X is alternatively denoted by σ_X^2 or $\mathrm{Var}(X)$.

- **EASIER FORMULA FOR VARIANCE:** $\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- **STANDARD DEVIATION OF A CONTINUOUS R.V.:** $\sigma_X := \sqrt{\mathbb{V}[X]}$
- **PROPERTIES OF EXPECTED VALUE & VARIANCE:** Let real numbers $a < b$. Then:

$$\mathbb{E}[aX + b] = a \cdot \mathbb{E}[X] + b \qquad \mathbb{V}[aX + b] = a^2 \cdot \mathbb{V}[X] \qquad \sigma_{aX+b} = |a| \cdot \sigma_X$$

- **PERCENTILE OF A CONTINUOUS RV:** Let X be continuous rv with pdf $f_X(x)$ & cdf $F_X(x)$ and $0 < p < 1$.

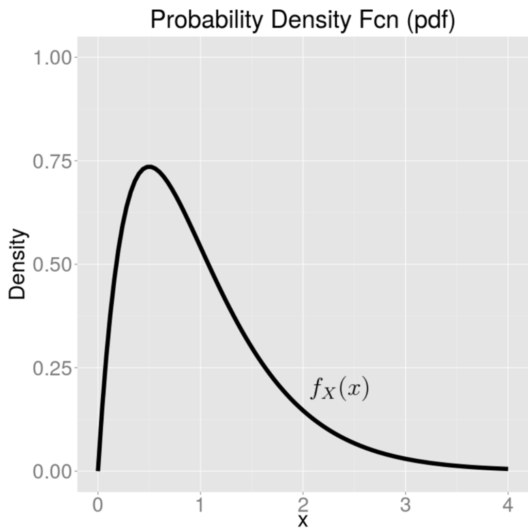
$$p = F_X(x_p) \iff p = \int_{-\infty}^{x_p} f_X(\xi)\ d\xi \iff p = \int_{-\infty}^{x_p} f_X(t)\ dt$$

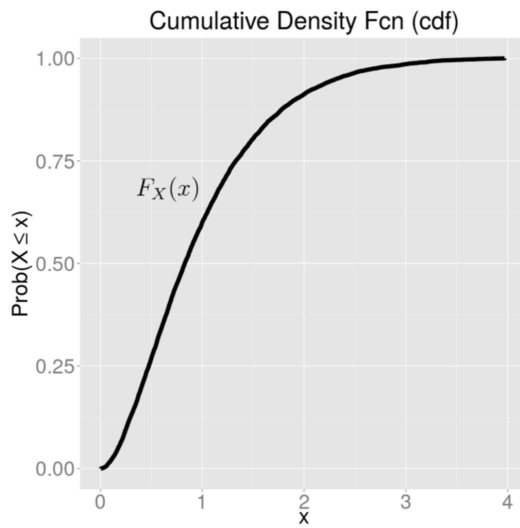
- **MEDIAN OF A CONTINUOUS R.V.:** Let X be continuous rv with pdf $f_X(x)$ & cdf $F_X(x)$. Then:

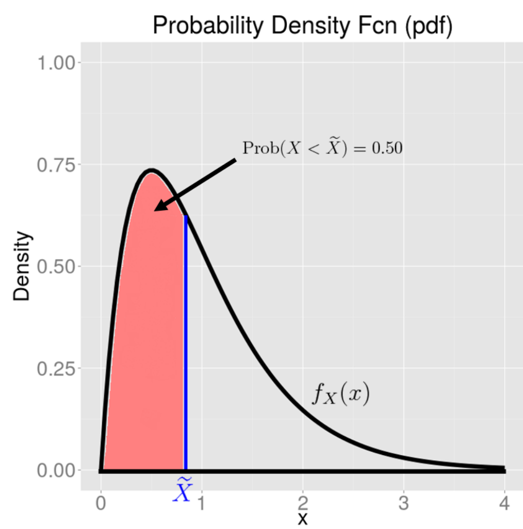
$$0.50 = F_X(x_{0.50}) \iff 0.50 = \int_{-\infty}^{x_{0.50}} f_X(\xi)\ d\xi \iff 0.50 = \int_{-\infty}^{x_{0.50}} f_X(t)\ dt$$

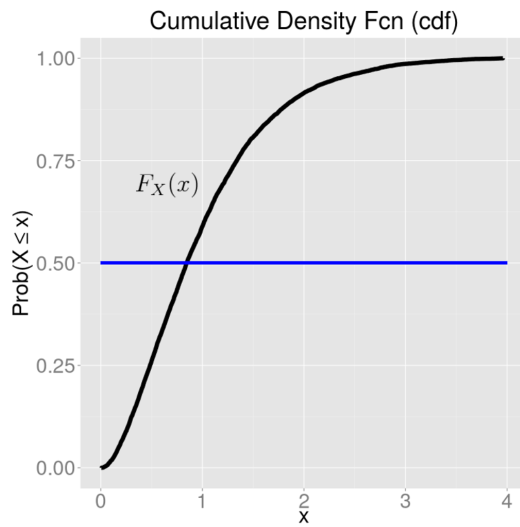
  NOTATION: The **median** of X is alternatively denoted by $\widetilde{X}$ or $\widetilde{\mu}_X$.

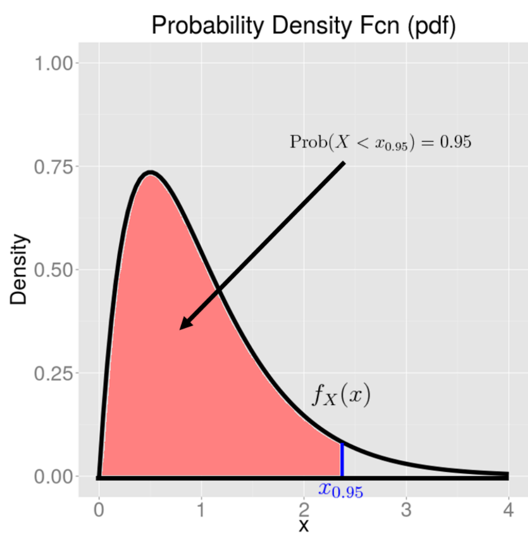
CONTINUOUS R.V.'S: PLOTS OF PDF, CDF, MEDIAN, PERCENTILE [DEVORE 4.2]

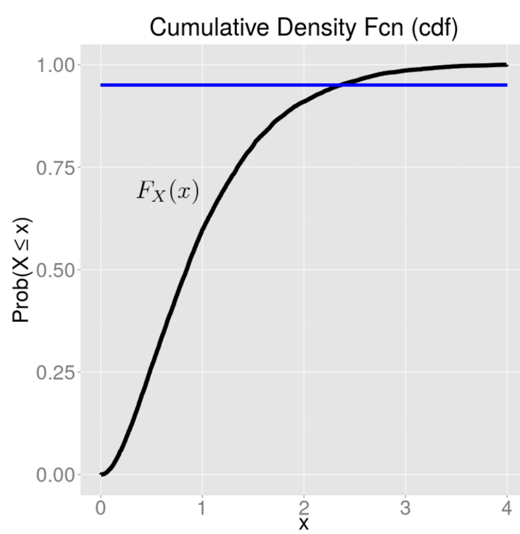












<u>NOTE:</u> Prob$(\cdots)$ means the same thing (probability) as $\mathbb{P}(\cdots)$.

EX 4.2.1: A local gas station has three pumps, each of which can pump up to 1000 gallons per week.
Let $X \equiv$ (Total amount of gas (in 1000's of gallons) pumped at the station in a week). Then, X has the pdf:

$$f_X(x) = \begin{cases} x/2 & \text{, if } 0 \le x < 1 \\ 1 - (5/16)x & \text{, if } 1 \le x \le 3 \\ 0 & \text{, otherwise} \end{cases}$$

(a) What is the support of random variable X?

(b) Determine the cdf of X, $F_X(x)$.

(c) **Using the cdf**, what is the probability that the station will pump at most 500 gallons in a week?

(d) **Using the cdf**, what is the probability that the station will pump between 1000 and 2000 gallons in a week?

(e) What is the expected amount of gas pumped by the station in a week?

(f) What is the variance of the amount of gas pumped by the station in a week?

(g) What is the standard deviation of the amount of gas pumped by the station in a week?

EX 4.2.2: Let $X \equiv$ (Monthly salary ($) where few workers earn very high monthly salaries). Then, the pdf of X is:

$$f_X(x) = \begin{cases} (3 \times 10^9)/x^4 & , \text{ if } x > 1000 \\ 0 & , \text{ if } x \leq 1000 \end{cases}$$

(a) What is the support of random variable X?

(b) Determine the cdf of X, $F_X(x)$.

(c) **Using the cdf**, what is the probability that a worker earns between $2000 and $3000 per month?

(d) What is the mean monthly salary, μ_X?

(e) What is the variance of the monthly salary, σ_X^2?

(f) What is the median monthly salary, $\widetilde{\mu}_X$?

(g) What is the probability that a worker's monthly salary is within 2 standard deviations of the mean value?

EX 4.2.3: Given the following cdf of X: $F_X(x) = \begin{cases} 0 & \text{, if } x \le 0 \\ (x/4)[1+\ln(4/x)] & \text{, if } 0 < x \le 4 \\ 1 & \text{, if } x > 4 \end{cases}$

What is the corresponding pdf of X, $f_X(x)$?

EX 4.2.4: Given the following cdf of X: $F_X(x) = e^{-e^{-x}}$ for $-\infty < x < \infty$

(a) Show that $\lim_{x \to -\infty} F_X(x) = 0$.

(b) Show that $\lim_{x \to \infty} F_X(x) = 1$.

(c) What is the corresponding pdf of X, $f_X(x)$?

(d) What is the median of X, $\widetilde{\mu}_X$?

(e) What is the 80th percentile of X, $x_{0.80}$?