EX 4.4.1: Suppose the lifetime (in hours) of a certain electrical component follows an Exponential($\lambda = 0.005$) distribution.

(a) What is the probability that the component lasts at most 300 hours?

Let $X \sim$ Exponential($\lambda = 0.005$). Then its cdf $F_X(x; \lambda = 0.005) = \begin{cases} 1 - e^{-0.005x} & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0 \end{cases}$

$\mathbb{P}(X \leq 300) = F_X(300; \lambda = 0.005) = 1 - e^{-(0.005)(300)} = 1 - e^{-1.5} \approx \boxed{\mathbf{0.77687}}$

(b) What is the probability that the component lasts at least 1000 hours?

$\mathbb{P}(X \geq 1000) = 1 - \mathbb{P}(X \leq 1000) = 1 - F_X(1000; \lambda = 0.005) = 1 - \left[1 - e^{-(0.005)(1000)}\right] = e^{-5} \approx \boxed{\mathbf{0.00674}}$

(c) What is the probability that the component lasts between 500 and 800 hours?

$$\begin{aligned}\mathbb{P}(500 \leq X \leq 800) &= \mathbb{P}(X \leq 800) - \mathbb{P}(X \leq 500) = F_X(800; \lambda = 0.005) - F_X(500; \lambda = 0.005) \\ &= \left[1 - e^{-(0.005)(800)}\right] - \left[1 - e^{-(0.005)(500)}\right] = e^{-2.5} - e^{-4} \approx \boxed{\mathbf{0.06377}}\end{aligned}$$

(d) What is the expected lifetime of the component?

$\mathbb{E}[X] = 1/\lambda = 1/0.005 = \boxed{\mathbf{200} \text{ hours}}$

(e) What is the variance of the component's lifetime?

$\mathbb{V}[X] = 1/\lambda^2 = 1/0.005^2 = \boxed{\mathbf{40000} \text{ hours}^2}$

(f) What is the standard deviation of the component's lifetime?

$\sigma_X = \sqrt{\mathbb{V}[X]} = \sqrt{40000} = \boxed{\mathbf{200} \text{ hours}}$

Notice that the standard deviation is equal to the mean.

(g) What is the probability that the component lasts at least 3000 hours given the component lasted at least 1000 hours?

METHOD 1: Apply the Memoryless Property

$$\begin{aligned}\mathbb{P}(X \geq 3000 \mid X \geq 1000) &= \mathbb{P}(X \geq (3000 - 1000)) = \mathbb{P}(X \geq 2000) = 1 - \mathbb{P}(X \leq 2000) \\ &= 1 - F_X(2000; \lambda = 0.005) = 1 - \left[1 - e^{-(0.005)(2000)}\right] = e^{-10} \approx \boxed{\mathbf{0.0000454}}\end{aligned}$$

METHOD 2: Apply the Definition of Conditional Probability

$$\begin{aligned}\mathbb{P}(X \geq 3000 \mid X \geq 1000) &= \frac{\mathbb{P}(X \geq 3000 \ \cap \ X \geq 1000)}{\mathbb{P}(X \geq 1000)} = \frac{\mathbb{P}(X \geq 3000)}{\mathbb{P}(X \geq 1000)} = \frac{1 - F_X(3000; \lambda = 0.005)}{1 - F_X(1000; \lambda = 0.005)} \\ &= \frac{1 - \left[1 - e^{-(0.005)(3000)}\right]}{1 - \left[1 - e^{-(0.005)(1000)}\right]} = \frac{e^{-15}}{e^{-5}} = e^{-10} \approx \boxed{\mathbf{0.0000454}}\end{aligned}$$

(h) What is the interquartile range of the component's lifetime?

Recall from Chapter 1 the definition of interquartile range: $x_{IQR} := x_{Q3} - x_{Q1} = x_{0.75} - x_{0.25}$

Find the 75^{th} percentile: $F_X(x_{0.75}; \lambda = 0.005) = 0.75 \implies 1 - e^{-0.005x_{0.75}} = 0.75 \implies x_{0.75} = \dfrac{\ln 0.25}{-0.005} \approx 277.2589$ hours

Find the 25^{th} percentile: $F_X(x_{0.25}; \lambda = 0.005) = 0.25 \implies 1 - e^{-0.005x_{0.25}} = 0.25 \implies x_{0.25} = \dfrac{\ln 0.75}{-0.005} \approx 57.5364$ hours

$\therefore \ x_{IQR} = x_{0.75} - x_{0.25} \approx 277.2589 - 57.5364 = \boxed{\mathbf{219.7225} \text{ hours}}$

EX 4.4.2: Let random variable $X \sim \text{Gamma}(\alpha = 9, \beta = 0.25)$.

(a) Compute $\mathbb{P}(X < 3)$.

$$\mathbb{P}(X < 3) = \mathbb{P}(X \leq 3) = \gamma(3/\beta; \alpha) = \gamma(3/0.25; \alpha = 9) = \gamma(12; \alpha = 9) \overset{LOOKUP}{\approx} \boxed{\mathbf{0.84497}}$$

LOOKUP means in this case find the entry in the Incomplete Gamma Function Table at row labeled 12 and column $\alpha = 9$.

(b) Compute $\mathbb{P}(X \geq 2.5)$.

$$\mathbb{P}(X \geq 2.5) = 1 - \mathbb{P}(X \leq 2.5) = 1 - \gamma(2.5/\beta; \alpha) = 1 - \gamma(2.5/0.25; \alpha = 9) = 1 - \gamma(10; \alpha = 9) \overset{LOOKUP}{\approx} 1 - 0.66718 = \boxed{\mathbf{0.33282}}$$

LOOKUP means in this case find the entry in the Incomplete Gamma Function Table at row labeled 10 and column $\alpha = 9$.

(c) Compute $\mathbb{P}(1 < X < 4.125)$.

$$\begin{aligned} \mathbb{P}(1 < X < 4.125) &= \mathbb{P}(X \leq 4.125) - \mathbb{P}(X \leq 1) = \gamma(4.125/\beta; \alpha) - \gamma(1/\beta; \alpha) = \gamma(4.125/0.25; \alpha = 9) - \gamma(1/0.25; \alpha = 9) \\ &= \gamma(16.5; \alpha = 9) - \gamma(4; \alpha = 9) \overset{LOOKUP}{\approx} 0.98331 - 0.02136 = \boxed{\mathbf{0.96195}} \end{aligned}$$

(d) What is the mean of X?

$$\mathbb{E}[X] = \alpha\beta = (9)(0.25) = \boxed{\mathbf{2.25}}$$

(e) What is the variance of X?

$$\mathbb{V}[X] = \alpha\beta^2 = (9)(0.25)^2 = \boxed{\mathbf{0.5625}}$$

(f) What is the standard deviation of X?

$$\sigma_X = \sqrt{\mathbb{V}[X]} = \sqrt{0.5625} = \boxed{\mathbf{0.75}}$$

(g) What is the 90^{th} percentile of X, $x_{0.90}$?
(Interpolate if necessary, which means find the closest entry in the table.)

$$\gamma(x_{0.90}/\beta; \alpha) = 0.90 \implies \gamma(x_{0.90}/0.25; \alpha = 9) = 0.90 \overset{(*)}{\implies} x_{0.90}/0.25 \approx 13 \implies x_{0.90} \approx (0.25)(13) = \boxed{\mathbf{3.25}}$$

$(*)$ REVERSE LOOKUP in the Incomplete Gamma Table along column $\alpha = 9$