

# EXPONENTIAL RANDOM VARIABLES [DEVORE 4.4]

• **EXPONENTIAL RANDOM VARIABLES:** Exponential rv's model lifetimes & inter-arrival waiting times:

– **Waiting times** between **consecutive arrivals** of a **Poisson process**:

- \* Time between radioactive decays of  $1\mu g$  of Iodine-123
- \* Time between emails an account received
- \* Time between insurance claims from a given demographic
- \* Time between industrial accidents at a factory

– **Distances** between **consecutive arrivals** of a **Poisson process**:

- \* Distance between mutations of a DNA strand

– **Lifetimes** of electrical components

– Concentrations of air pollutants in a major city

• **MEMORYLESS PROPERTY OF EXPONENTIAL RANDOM VARIABLE'S:** Let  $X \sim \text{Exponential}(\lambda)$ .

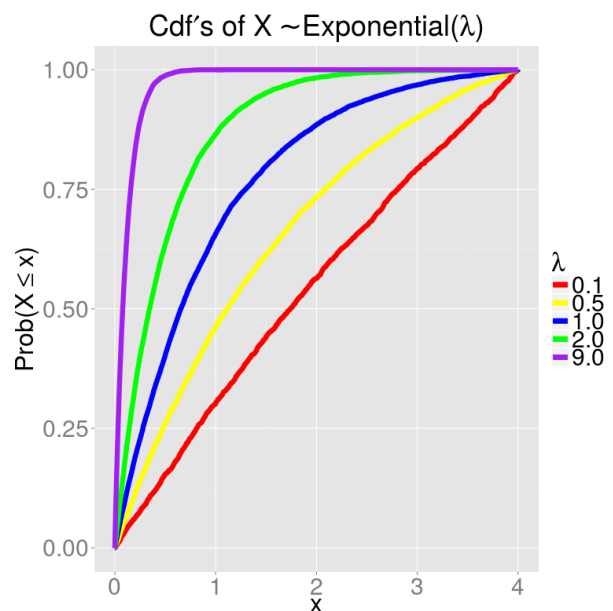
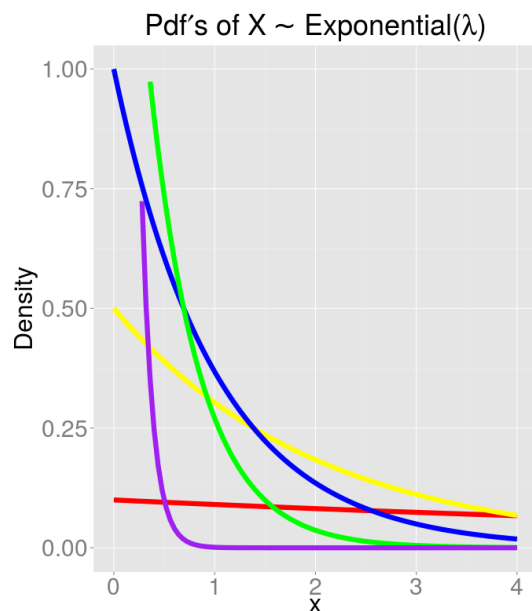
Then,  $\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$

e.g.  $\mathbb{P}(X > 50 \mid X > 20) = \mathbb{P}(X > 30)$

• **EXPONENTIAL RANDOM VARIABLES (SUMMARY):**

Notation	$X \sim \text{Exponential}(\lambda), \lambda > 0$
Parameter(s)	$\lambda \equiv \text{Arrival Rate} (= \alpha \text{ of Poisson Process})$
Support	$\text{Supp}(X) = [0, \infty)$
pdf	$f_X(x; \lambda) = \lambda e^{-\lambda x}$
cdf	$F_X(x; \lambda) = \begin{cases} 1 - e^{-\lambda x} & , \text{ if } x \geq 0 \\ 0 & , \text{ if } x < 0 \end{cases}$
Mean	$\mathbb{E}[X] = 1/\lambda$
Variance	$\mathbb{V}[X] = 1/\lambda^2$

• **EXPONENTIAL DENSITY PLOTS (PDF's & CDF's):**



# GAMMA RANDOM VARIABLES [DEVORE 4.4]

• **GAMMA RANDOM VARIABLES:** Gamma random variables reasonably model:

- Waiting time until  $\alpha$  arrivals of Poisson process occur (if  $\alpha$  is an **integer**)
- Processes that would typically be modeled by Exponential rv's except...
  - \* ...the pdf is **unimodal & skewed**
  - \* ...the **memoryless property** is not realized

In fact, an Exponential( $\lambda$ ) rv is equivalently a Gamma( $\alpha = 1, \beta = 1/\lambda$ ) rv.

• **GAMMA FUNCTION (DEFINITION):**  $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \text{where } \alpha > 0$

• **GAMMA FUNCTION (PROPERTIES):**

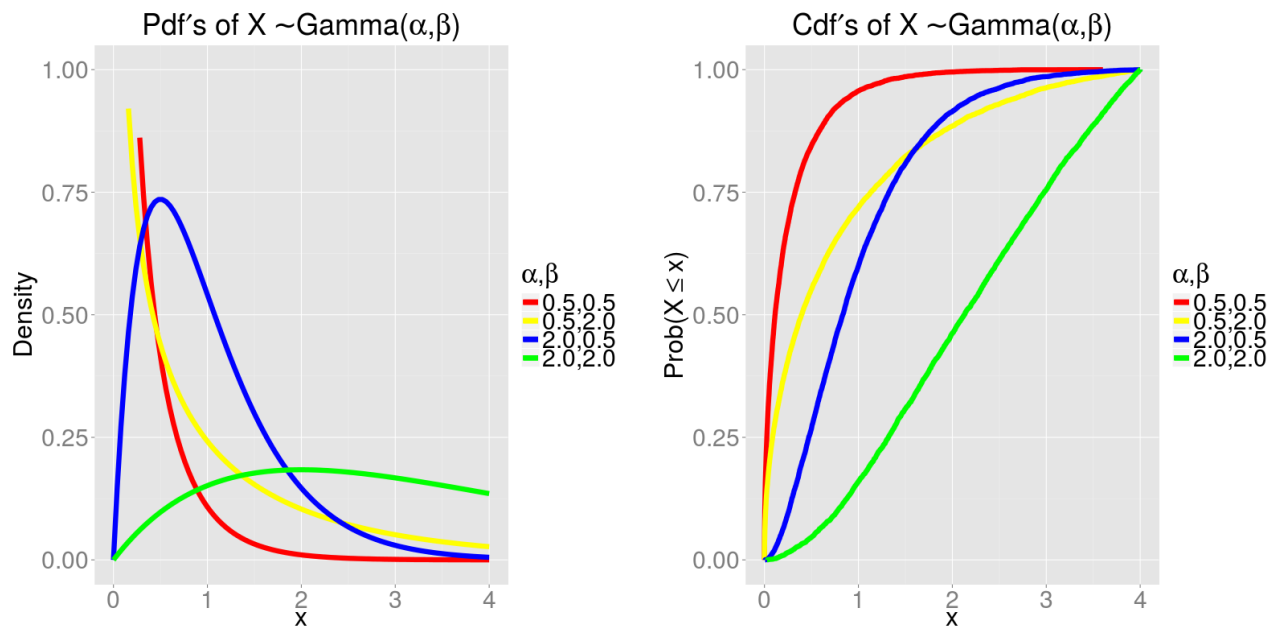
$\Gamma(n)$	$= (n-1)!$	where $n$ is a <b>positive integer</b>
$\Gamma(\alpha+1)$	$= \alpha\Gamma(\alpha)$	where $\alpha > 0$

• **GAMMA RANDOM VARIABLES (SUMMARY):**

Notation	$X \sim \text{Gamma}(\alpha, \beta), \quad \alpha, \beta > 0$
Parameter(s)	$\alpha \equiv$ Shape parameter
	$\beta \equiv$ Scale parameter
Support	$\text{Supp}(X) = [0, \infty)$
pdf	$f_X(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$
cdf	$\gamma(x/\beta; \alpha)$
Mean	$\mathbb{E}[X] = \alpha\beta$
Variance	$\mathbb{V}[X] = \alpha\beta^2$

A Gamma( $\alpha = 1, \beta = 1/\lambda$ ) rv is identical to a Exponential( $\lambda$ ) rv.

• **GAMMA DENSITY PLOTS (PDF's & CDF's):**



INCOMPLETE GAMMA FUNCTION  $\gamma(x; \alpha) := \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt$

	Shape Parameter ( $\alpha$ )									
$x$	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
<b>0.5</b>	0.68269	0.39347	0.19875	0.09020	0.03743	0.01439	0.00517	0.00175	0.00056	0.00017
<b>1</b>	0.84270	0.63212	0.42759	0.26424	0.15085	0.08030	0.04016	0.01899	0.00853	0.00366
<b>1.5</b>	0.91674	0.77687	0.60837	0.44217	0.30001	0.19115	0.11500	0.06564	0.03571	0.01858
<b>2</b>	0.95450	0.86466	0.73854	0.59399	0.45058	0.32332	0.22022	0.14288	0.08859	0.05265
<b>2.5</b>	0.97465	0.91792	0.82820	0.71270	0.58412	0.45619	0.34004	0.24242	0.16569	0.10882
<b>3</b>	0.98569	0.95021	0.88839	0.80085	0.69378	0.57681	0.46025	0.35277	0.26008	0.18474
<b>3.5</b>	0.99185	0.96980	0.92810	0.86411	0.77936	0.67915	0.57112	0.46337	0.36288	0.27456
<b>4</b>	0.99532	0.98168	0.95399	0.90842	0.84376	0.76190	0.66741	0.56653	0.46585	0.37116
<b>4.5</b>	0.99730	0.98889	0.97071	0.93890	0.89094	0.82642	0.74734	0.65770	0.56273	0.46790
<b>5</b>	0.99843	0.99326	0.98143	0.95957	0.92476	0.87535	0.81143	0.73497	0.64951	0.55951
<b>5.5</b>	0.99909	0.99591	0.98827	0.97344	0.94862	0.91162	0.86138	0.79830	0.72429	0.64248
<b>6</b>	0.99947	0.99752	0.99262	0.98265	0.96521	0.93803	0.89944	0.84880	0.78669	0.71494
<b>6.5</b>	0.99969	0.99850	0.99536	0.98872	0.97662	0.95696	0.92789	0.88815	0.83739	0.77633
<b>7</b>	0.99982	0.99909	0.99709	0.99270	0.98439	0.97036	0.94882	0.91823	0.87767	0.82701
<b>7.5</b>	0.99989	0.99945	0.99818	0.99530	0.98964	0.97974	0.96400	0.94085	0.90906	0.86794
<b>8</b>	0.99994	0.99966	0.99887	0.99698	0.99316	0.98625	0.97488	0.95762	0.93312	0.90037
<b>8.5</b>	0.99996	0.99980	0.99929	0.99807	0.99550	0.99072	0.98260	0.96989	0.95128	0.92564
<b>9</b>	0.99998	0.99988	0.99956	0.99877	0.99705	0.99377	0.98803	0.97877	0.96483	0.94504
<b>9.5</b>	0.99999	0.99993	0.99973	0.99921	0.99808	0.99584	0.99181	0.98514	0.97481	0.95974
<b>10</b>	0.99999	0.99995	0.99983	0.99950	0.99875	0.99723	0.99443	0.98966	0.98209	0.97075
<b>10.5</b>	1.00000	0.99997	0.99989	0.99968	0.99919	0.99817	0.99623	0.99285	0.98735	0.97891
<b>11</b>	1.00000	0.99998	0.99993	0.99980	0.99948	0.99879	0.99746	0.99508	0.99112	0.98490
<b>11.5</b>	1.00000	0.99999	0.99996	0.99987	0.99966	0.99920	0.99830	0.99664	0.99380	0.98925
<b>12</b>	1.00000	0.99999	0.99998	0.99992	0.99978	0.99948	0.99886	0.99771	0.99570	0.99240
<b>12.5</b>	1.00000	1.00000	0.99998	0.99995	0.99986	0.99966	0.99924	0.99845	0.99703	0.99465
<b>13</b>	1.00000	1.00000	0.99999	0.99997	0.99991	0.99978	0.99950	0.99895	0.99796	0.99626
<b>13.5</b>	1.00000	1.00000	0.99999	0.99998	0.99994	0.99986	0.99967	0.99929	0.99860	0.99740
<b>14</b>	1.00000	1.00000	1.00000	0.99999	0.99996	0.99991	0.99978	0.99953	0.99905	0.99819
<b>14.5</b>	1.00000	1.00000	1.00000	0.99999	0.99998	0.99994	0.99986	0.99968	0.99935	0.99875
<b>15</b>	1.00000	1.00000	1.00000	1.00000	0.99999	0.99996	0.99991	0.99979	0.99956	0.99914
<b>15.5</b>	1.00000	1.00000	1.00000	1.00000	0.99999	0.99997	0.99994	0.99986	0.99970	0.99941
<b>16</b>	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998	0.99996	0.99991	0.99980	0.99960
<b>16.5</b>	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99997	0.99994	0.99987	0.99973
<b>17</b>	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998	0.99996	0.99991	0.99982
<b>17.5</b>	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99997	0.99994	0.99988
<b>18</b>	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998	0.99996	0.99992
<b>18.5</b>	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99997	0.99994
<b>19</b>	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998	0.99996
<b>19.5</b>	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99997
<b>20</b>	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998

INCOMPLETE GAMMA FUNCTION  $\gamma(x; \alpha) := \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt$

	Shape Parameter ( $\alpha$ )									
$x$	5.5	6	6.5	7	7.5	8	8.5	9	9.5	10
<b>0.5</b>	0.00005	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
<b>1</b>	0.00150	0.00059	0.00023	0.00008	0.00003	0.00001	0.00000	0.00000	0.00000	0.00000
<b>1.5</b>	0.00927	0.00446	0.00207	0.00093	0.00040	0.00017	0.00007	0.00003	0.00001	0.00000
<b>2</b>	0.03008	0.01656	0.00881	0.00453	0.00226	0.00110	0.00052	0.00024	0.00011	0.00005
<b>2.5</b>	0.06883	0.04202	0.02481	0.01419	0.00787	0.00425	0.00223	0.00114	0.00057	0.00028
<b>3</b>	0.12664	0.08392	0.05385	0.03351	0.02025	0.01190	0.00681	0.00380	0.00207	0.00110
<b>3.5</b>	0.20092	0.14239	0.09785	0.06529	0.04235	0.02674	0.01645	0.00987	0.00579	0.00331
<b>4</b>	0.28670	0.21487	0.15640	0.11067	0.07622	0.05113	0.03345	0.02136	0.01333	0.00813
<b>4.5</b>	0.37811	0.29707	0.22706	0.16895	0.12248	0.08659	0.05974	0.04026	0.02652	0.01709
<b>5</b>	0.46961	0.38404	0.30607	0.23782	0.18026	0.13337	0.09639	0.06809	0.04705	0.03183
<b>5.5</b>	0.55674	0.47108	0.38918	0.31396	0.24741	0.19051	0.14344	0.10564	0.07616	0.05378
<b>6</b>	0.63636	0.55432	0.47236	0.39370	0.32097	0.25602	0.19986	0.15276	0.11437	0.08392
<b>6.5</b>	0.70667	0.63096	0.55219	0.47348	0.39770	0.32724	0.26381	0.20843	0.16143	0.12262
<b>7</b>	0.76701	0.69929	0.62616	0.55029	0.47447	0.40129	0.33290	0.27091	0.21631	0.16950
<b>7.5</b>	0.81750	0.75856	0.69265	0.62185	0.54858	0.47536	0.40452	0.33803	0.27740	0.22359
<b>8</b>	0.85887	0.80876	0.75087	0.68663	0.61795	0.54704	0.47617	0.40745	0.34272	0.28338
<b>8.5</b>	0.89212	0.85040	0.80070	0.74382	0.68114	0.61440	0.54563	0.47689	0.41013	0.34703
<b>9</b>	0.91842	0.88431	0.84248	0.79322	0.73733	0.67610	0.61116	0.54435	0.47756	0.41259
<b>9.5</b>	0.93891	0.91147	0.87690	0.83505	0.78627	0.73134	0.67147	0.60818	0.54316	0.47817
<b>10</b>	0.95466	0.93291	0.90479	0.86986	0.82807	0.77978	0.72577	0.66718	0.60542	0.54207
<b>10.5</b>	0.96663	0.94962	0.92707	0.89837	0.86317	0.82149	0.77371	0.72059	0.66320	0.60287
<b>11</b>	0.97563	0.96248	0.94464	0.92139	0.89220	0.85681	0.81528	0.76801	0.71574	0.65949
<b>11.5</b>	0.98232	0.97227	0.95832	0.93973	0.91586	0.88627	0.85075	0.80941	0.76266	0.71121
<b>12</b>	0.98727	0.97966	0.96887	0.95418	0.93491	0.91050	0.88057	0.84497	0.80385	0.75761
<b>12.5</b>	0.99088	0.98518	0.97692	0.96543	0.95006	0.93017	0.90529	0.87508	0.83946	0.79857
<b>13</b>	0.99351	0.98927	0.98300	0.97411	0.96198	0.94597	0.92554	0.90024	0.86981	0.83419
<b>13.5</b>	0.99540	0.99227	0.98756	0.98075	0.97126	0.95852	0.94193	0.92100	0.89535	0.86474
<b>14</b>	0.99676	0.99447	0.99095	0.98577	0.97843	0.96838	0.95506	0.93794	0.91657	0.89060
<b>14.5</b>	0.99773	0.99606	0.99345	0.98955	0.98392	0.97606	0.96547	0.95162	0.93401	0.91224
<b>15</b>	0.99842	0.99721	0.99529	0.99237	0.98808	0.98200	0.97365	0.96255	0.94820	0.93015
<b>15.5</b>	0.99890	0.99803	0.99663	0.99446	0.99122	0.98654	0.98003	0.97121	0.95963	0.94481
<b>16</b>	0.99924	0.99862	0.99760	0.99599	0.99356	0.99000	0.98495	0.97801	0.96875	0.95670
<b>16.5</b>	0.99947	0.99903	0.99830	0.99712	0.99531	0.99261	0.98873	0.98331	0.97596	0.96626
<b>17</b>	0.99964	0.99933	0.99880	0.99794	0.99659	0.99457	0.99160	0.98740	0.98162	0.97388
<b>17.5</b>	0.99975	0.99953	0.99915	0.99853	0.99754	0.99603	0.99378	0.99055	0.98603	0.97990
<b>18</b>	0.99983	0.99968	0.99941	0.99896	0.99823	0.99711	0.99541	0.99294	0.98944	0.98462
<b>18.5</b>	0.99988	0.99978	0.99959	0.99926	0.99873	0.99790	0.99663	0.99476	0.99207	0.98830
<b>19</b>	0.99992	0.99985	0.99971	0.99948	0.99910	0.99849	0.99754	0.99613	0.99407	0.99114
<b>19.5</b>	0.99995	0.99989	0.99980	0.99964	0.99936	0.99891	0.99821	0.99715	0.99558	0.99333
<b>20</b>	0.99996	0.99993	0.99986	0.99974	0.99955	0.99922	0.99871	0.99791	0.99673	0.99500



**EX 4.4.2:** Let random variable  $X \sim \text{Gamma}(\alpha = 9, \beta = 0.25)$ .

(a) Compute  $\mathbb{P}(X < 3)$ .

(b) Compute  $\mathbb{P}(X \geq 2.5)$ .

(c) Compute  $\mathbb{P}(1 < X < 4.125)$ .

(d) What is the mean of  $X$ ?

(e) What is the variance of  $X$ ?

(f) What is the standard deviation of  $X$ ?

(g) What is the 90<sup>th</sup> percentile of  $X$ ,  $x_{0.90}$ ?

(Interpolate if necessary, which means find the closest entry in the table.)