EX 5.3.1: Let size ($n = 2$) random sample $X_1, X_2 \overset{iid}{\sim}$ pmf $p_X(k)$ such that:

k	0	1	2
$p_X(k)$	0.65	0.25	0.10

(a) What is the population mean μ and population variance σ^2?

$$\mu \equiv \mathbb{E}[X] = \sum_{k \in \text{Supp}(X)} k \cdot p_X(k) = (0)(0.65) + (1)(0.25) + (2)(0.10) = \boxed{\mathbf{0.45}}$$

$$\mathbb{E}[X^2] = \sum_{k \in \text{Supp}(X)} k^2 \cdot p_X(k) = (0^2)(0.65) + (1^2)(0.25) + (2^2)(0.10) = 0.65$$

$$\therefore \quad \sigma^2 \equiv \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 0.65 - (0.45)^2 = \boxed{\mathbf{0.4475}}$$

(b) Construct the sampling distributions for the sample mean $\overline{X}$ and the sample variance S^2.

$X_1 = j_1$	$X_2 = j_2$	$\mathbb{P}(X_1 = j_1 \cap X_2 = j_2)$	$\overline{X} = (X_1 + X_2)/2$	$S^2 = \frac{1}{n-1}\left[(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2\right]$
0	0	$p_X(0) \cdot p_X(0) = (0.65)(0.65) = 0.4225$	$(0+0)/2 = 0$	$(0-0)^2 + (0-0)^2 = 0$
0	1	$p_X(0) \cdot p_X(1) = (0.65)(0.25) = 0.1625$	$(0+1)/2 = 0.5$	$(0-0.5)^2 + (1-0.5)^2 = 0.5$
0	2	$p_X(0) \cdot p_X(2) = (0.65)(0.10) = 0.0650$	$(0+2)/2 = 1$	$(0-1)^2 + (2-1)^2 = 2$
1	0	$p_X(1) \cdot p_X(0) = (0.25)(0.65) = 0.1625$	$(1+0)/2 = 0.5$	$(1-0.5)^2 + (0-0.5)^2 = 0.5$
1	1	$p_X(1) \cdot p_X(1) = (0.25)(0.25) = 0.0625$	$(1+1)/2 = 1$	$(1-1)^2 + (1-1)^2 = 0$
1	2	$p_X(1) \cdot p_X(2) = (0.25)(0.10) = 0.0250$	$(1+2)/2 = 1.5$	$(1-1.5)^2 + (2-1.5)^2 = 0.5$
2	0	$p_X(2) \cdot p_X(0) = (0.10)(0.65) = 0.0650$	$(2+0)/2 = 1$	$(2-1)^2 + (0-1)^2 = 2$
2	1	$p_X(2) \cdot p_X(1) = (0.10)(0.25) = 0.0250$	$(2+1)/2 = 1.5$	$(2-1.5)^2 + (1-1.5)^2 = 0.5$
2	2	$p_X(2) \cdot p_X(2) = (0.10)(0.10) = 0.0100$	$(2+2)/2 = 2$	$(2-2)^2 + (2-2)^2 = 0$

$\Longrightarrow$

k	0	0.5	1	1.5	2
$p_{\overline{X}}(k)$	0.4225	0.1625 + 0.1625	0.0650 + 0.0625 + 0.0650	0.0250 + 0.0250	0.0100

$\therefore$ The sampling distribution for $\overline{X}$ is the pmf

k	0	0.5	1	1.5	2
$p_{\overline{X}}(k)$	0.4225	0.3250	0.1925	0.0500	0.0100

SANITY CHECK: $\displaystyle\sum_{k \in \text{Supp}(\overline{X})} p_{\overline{X}}(k) = 0.4225 + 0.3250 + 0.1925 + 0.05 + 0.01 = 1$ ✓

$\Longrightarrow$

k	0	0.5	2
$p_{S^2}(k)$	0.4225 + 0.0625 + 0.0100	0.1625 + 0.1625 + 0.0250 + 0.0250	0.0650 + 0.0650

$\therefore$ The sampling distribution for S^2 is the pmf

k	0	0.5	2
$p_{S^2}(k)$	0.4950	0.3750	0.1300

SANITY CHECK: $\displaystyle\sum_{k \in \text{Supp}(S^2)} p_{S^2}(k) = 0.4950 + 0.3750 + 0.13 = 1$ ✓

(c) What is the expected value & variance of the sample mean, $\mu_{\overline{X}}$ & $\sigma^2_{\overline{X}}$?

$$\mu_{\overline{X}} \equiv \mathbb{E}[\overline{X}] = \sum_{k \in \text{Supp}(\overline{X})} k \cdot p_{\overline{X}}(k) = (0)(0.4225) + (0.5)(0.3250) + (1)(0.1925) + (1.5)(0.05) + (2)(0.01) = \boxed{\mathbf{0.45}}$$

$$\mathbb{E}[\overline{X}^2] = \sum_{k \in \text{Supp}(\overline{X})} k^2 \cdot p_{\overline{X}}(k) = (0^2)(0.4225) + (0.5^2)(0.3250) + (1^2)(0.1925) + (1.5^2)(0.05) + (2^2)(0.01) = 0.42625$$

$$\therefore \quad \sigma^2_{\overline{X}} \equiv \mathbb{V}[\overline{X}] = \mathbb{E}[\overline{X}^2] - (\mathbb{E}[\overline{X}])^2 = 0.42625 - (0.45)^2 = \boxed{\mathbf{0.22375}}$$

(d) What is the expected value & variance of the sample variance, μ_{S^2} & $\sigma^2_{S^2}$?

$$\mu_{S^2} \equiv \mathbb{E}[S^2] = \sum_{k \in \text{Supp}(S^2)} k \cdot p_{S^2}(k) = (0)(0.4950) + (0.5)(0.3750) + (2)(0.13) = \boxed{\mathbf{0.4475}}$$

$$\mathbb{E}[(S^2)^2] = \sum_{k \in \text{Supp}(S^2)} k^2 \cdot p_{S^2}(k) = (0^2)(0.4950) + (0.5^2)(0.3750) + (2^2)(0.13) = 0.61375$$

$$\therefore \quad \sigma^2_{S^2} \equiv \mathbb{V}[S^2] = \mathbb{E}[(S^2)^2] - (\mathbb{E}[S^2])^2 = 0.61375 - (0.4475)^2 = \boxed{\mathbf{0.41349375}}$$

EX 5.3.2: Let size ($n = 3$) random sample $Y_1, Y_2, Y_3 \overset{iid}{\sim}$ pmf $p_Y(k)$ such that:

k	2	3
$p_Y(k)$	0.2	0.8

(a) What is the population mean μ and population variance σ^2?

$$\mu \equiv \mathbb{E}[Y] = \sum_{k \in \text{Supp}(Y)} k \cdot p_Y(k) = (2)(0.2) + (3)(0.8) = \boxed{\mathbf{2.8}}$$

$$\mathbb{E}[Y^2] = \sum_{k \in \text{Supp}(Y)} k^2 \cdot p_Y(k) = (2^2)(0.2) + (3^2)(0.8) = 8$$

$$\therefore \quad \sigma^2 \equiv \mathbb{V}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 8 - (2.8)^2 = \boxed{\mathbf{0.16}}$$

(b) Construct the sampling distributions for the sample mean $\overline{Y}$ and the sample total $Y_1 + Y_2 + Y_3$.

$Y_1 = j_1$	$Y_2 = j_2$	$Y_3 = j_3$	$\mathbb{P}(Y_1 = j_1 \ \cap \ Y_2 = j_2 \ \cap \ Y_3 = j_3)$	$Y_1 + Y_2 + Y_3$	$\overline{Y} = (Y_1 + Y_2 + Y_3)/3$
2	2	2	$p_Y(2) \cdot p_Y(2) \cdot p_Y(2) = (0.2)(0.2)(0.2) = 0.008$	$2+2+2=6$	6/3
2	2	3	$p_Y(2) \cdot p_Y(2) \cdot p_Y(3) = (0.2)(0.2)(0.8) = 0.032$	$2+2+3=7$	7/3
2	3	2	$p_Y(2) \cdot p_Y(3) \cdot p_Y(2) = (0.2)(0.8)(0.2) = 0.032$	$2+3+2=7$	7/3
2	3	3	$p_Y(2) \cdot p_Y(3) \cdot p_Y(3) = (0.2)(0.8)(0.8) = 0.128$	$2+3+3=8$	8/3
3	2	2	$p_Y(3) \cdot p_Y(2) \cdot p_Y(2) = (0.8)(0.2)(0.2) = 0.032$	$3+2+2=7$	7/3
3	2	3	$p_Y(3) \cdot p_Y(2) \cdot p_Y(3) = (0.8)(0.2)(0.8) = 0.128$	$3+2+3=8$	8/3
3	3	2	$p_Y(3) \cdot p_Y(3) \cdot p_Y(2) = (0.8)(0.8)(0.2) = 0.128$	$3+3+2=8$	8/3
3	3	3	$p_Y(3) \cdot p_Y(3) \cdot p_Y(3) = (0.8)(0.8)(0.8) = 0.512$	$3+3+3=9$	9/3

$\Longrightarrow$

k	6/3	7/3	8/3	9/3
$p_{\overline{Y}}(k)$	0.008	$0.032 + 0.032 + 0.032$	$0.128 + 0.128 + 0.128$	0.512

$\therefore$ The sampling distribution for $\overline{Y}$ is the pmf

k	6/3	7/3	8/3	9/3
$p_{\overline{Y}}(k)$	0.008	0.096	0.384	0.512

SANITY CHECK: $\displaystyle\sum_{k \in \text{Supp}(\overline{Y})} p_{\overline{Y}}(k) = 0.008 + 0.096 + 0.384 + 0.512 = 1$ ✓

$\Longrightarrow$

k	6	7	8	9
$p_{Y_1+Y_2+Y_3}(k)$	0.008	$0.032 + 0.032 + 0.032$	$0.128 + 0.128 + 0.128$	0.512

$\therefore$ The sampling distribution for $Y_1 + Y_2 + Y_3$ is the pmf

k	6	7	8	9
$p_{Y_1+Y_2+Y_3}(k)$	0.008	0.096	0.384	0.512

SANITY CHECK: $\displaystyle\sum_{k \in \text{Supp}(Y_1+Y_2+Y_3)} p_{Y_1+Y_2+Y_3}(k) = 0.008 + 0.096 + 0.384 + 0.512 = 1$ ✓

(c) What is the expected value & variance of the sample mean, $\mu_{\overline{Y}}$ & $\sigma^2_{\overline{Y}}$?

$$\mu_{\overline{Y}} \equiv \mathbb{E}[\overline{Y}] = \sum_{k \in \text{Supp}(\overline{Y})} k \cdot p_{\overline{Y}}(k) = (6/3)(0.008) + (7/3)(0.096) + (8/3)(0.384) + (9/3)(0.512) = \boxed{\mathbf{2.8}}$$

$$\mathbb{E}[\overline{Y}^2] = \sum_{k \in \text{Supp}(\overline{Y})} k^2 \cdot p_{\overline{Y}}(k) = (6/3)^2(0.008) + (7/3)^2(0.096) + (8/3)^2(0.384) + (9/3)^2(0.512) = 7.893333333$$

$$\therefore \quad \sigma^2_{\overline{Y}} \equiv \mathbb{V}[\overline{Y}] = \mathbb{E}[\overline{Y}^2] - (\mathbb{E}[\overline{Y}])^2 = 7.893333333 - (2.8)^2 = \boxed{\mathbf{0.053333333}}$$

(d) What is the expected value & variance of the sample total, $\mu_{Y_1+Y_2+Y_3}$ & $\sigma^2_{Y_1+Y_2+Y_3}$?

$$\mu_{Y_1+Y_2+Y_3} \equiv \mathbb{E}[Y_1 + Y_2 + Y_3] = (6)(0.008) + (7)(0.096) + (8)(0.384) + (9)(0.512) = \boxed{\mathbf{8.4}}$$

$$\mathbb{E}[(Y_1 + Y_2 + Y_3)^2] = (6)^2(0.008) + (7)^2(0.096) + (8)^2(0.384) + (9)^2(0.512) = 71.04$$

$$\therefore \quad \sigma^2_{Y_1+Y_2+Y_3} \equiv \mathbb{V}[Y_1 + Y_2 + Y_3] = \mathbb{E}[(Y_1 + Y_2 + Y_3)^2] - (\mathbb{E}[Y_1 + Y_2 + Y_3])^2 = 71.04 - (8.4)^2 = \boxed{\mathbf{0.48}}$$

EX 5.3.3: Let size $(n = 2)$ random sample $W_1, W_2 \overset{iid}{\sim}$ Binomial$(2, 0.35)$.

(a) What is the population mean μ and population variance σ^2?

$\mu = (2)(0.35) = \boxed{\mathbf{0.7}}$ $\quad \sigma^2 = (2)(0.35)(1-0.35) = \boxed{\mathbf{0.455}}$ $\quad p_W(k) = \binom{2}{k} 0.35^k 0.65^{2-k}$

$p_W(0) = \binom{2}{0} 0.35^0 0.65^2 = 0.4225$ $\quad p_W(1) = \binom{2}{1} 0.35^1 0.65^1 = 0.4550$ $\quad p_W(2) = \binom{2}{2} 0.35^2 0.65^0 = 0.1225$

(b) Construct the sampling distributions for the sample mean $\overline{W}$ and the sample minimum $W_{(1)}$.

$W_1 = j_1$	$W_2 = j_2$	$\mathbb{P}(W_1 = j_1 \ \cap \ W_2 = j_2)$	$\overline{W} = (W_1 + W_2)/2$	$W_{(1)} = \min\{W_1, W_2\}$
0	0	$p_W(0) \cdot p_W(0) = (0.4225)(0.4225) = 0.17850625$	$(0+0)/2 = 0$	$\min\{0,0\} = 0$
0	1	$p_W(0) \cdot p_W(1) = (0.4225)(0.4550) = 0.19223750$	$(0+1)/2 = 0.5$	$\min\{0,1\} = 0$
0	2	$p_W(0) \cdot p_W(2) = (0.4225)(0.1225) = 0.05175625$	$(0+2)/2 = 1$	$\min\{0,2\} = 0$
1	0	$p_W(1) \cdot p_W(0) = (0.4550)(0.4225) = 0.19223750$	$(1+0)/2 = 0.5$	$\min\{1,0\} = 0$
1	1	$p_W(1) \cdot p_W(1) = (0.4550)(0.4550) = 0.20702500$	$(1+1)/2 = 1$	$\min\{1,1\} = 1$
1	2	$p_W(1) \cdot p_W(2) = (0.4550)(0.1225) = 0.05573750$	$(1+2)/2 = 1.5$	$\min\{1,2\} = 1$
2	0	$p_W(2) \cdot p_W(0) = (0.1225)(0.4225) = 0.05175625$	$(2+0)/2 = 1$	$\min\{2,0\} = 0$
2	1	$p_W(2) \cdot p_W(1) = (0.1225)(0.4550) = 0.05573750$	$(2+1)/2 = 1.5$	$\min\{2,1\} = 1$
2	2	$p_W(2) \cdot p_W(2) = (0.1225)(0.1225) = 0.01500625$	$(2+2)/2 = 2$	$\min\{2,2\} = 2$

$\Longrightarrow$

k	0	0.5	1	1.5	2
$p_{\overline{W}}(k)$	0.17850625	0.19223750 + 0.19223750	0.05175625 + 0.20702500 + 0.05175625	0.05573750 + 0.05573750	0.01500625

$\therefore$ The sampling distribution for $\overline{W}$ is the pmf

k	0	0.5	1	1.5	2
$p_{\overline{W}}(k)$	0.17850625	0.384475	0.3105375	0.111475	0.01500625

SANITY CHECK: $\displaystyle\sum_{k \in \text{Supp}(\overline{W})} p_{\overline{W}}(k) = 0.17850625 + 0.384475 + 0.3105375 + 0.111475 + 0.01500625 = 1$ ✓

$\Longrightarrow$

k	0	1	2
$p_{W_{(1)}}(k)$	0.17850625 + 0.19223750 + 0.05175625 + 0.19223750 + 0.05175625	0.20702500 + 0.05573750 + 0.05573750	0.01500625

$\therefore$ The sampling distribution for $W_{(1)}$ is the pmf

k	0	1	2
$p_{W_{(1)}}(k)$	0.66649375	0.3185	0.01500625

SANITY CHECK: $\displaystyle\sum_{k \in \text{Supp}(W_{(1)})} p_{W_{(1)}}(k) = 0.66649375 + 0.3185 + 0.01500625 = 1$ ✓

(c) What is the expected value & variance of the sample mean, $\mu_{\overline{W}}$ & $\sigma^2_{\overline{W}}$?

$\mu_{\overline{W}} \equiv \mathbb{E}[\overline{W}] = (0)(0.17850625) + (0.5)(0.384475) + (1)(0.3105375) + (1.5)(0.111475) + (2)(0.01500625) = \boxed{\mathbf{0.7}}$

$\mathbb{E}[\overline{W}^2] = (0)^2(0.17850625) + (0.5)^2(0.384475) + (1)^2(0.3105375) + (1.5)^2(0.111475) + (2)^2(0.01500625) = 0.7175$

$\therefore \quad \sigma^2_{\overline{W}} \equiv \mathbb{V}[\overline{W}] = \mathbb{E}[\overline{W}^2] - (\mathbb{E}[\overline{W}])^2 = 0.7175 - (0.7)^2 = \boxed{\mathbf{0.2275}}$

(d) What is the expected value & variance of the sample total, $\mu_{W_{(1)}}$ & $\sigma^2_{W_{(1)}}$?

$\mu_{W_{(1)}} \equiv \mathbb{E}[W_{(1)}] = (0)(0.66649375) + (1)(0.3185) + (2)(0.01500625) = \boxed{\mathbf{0.3485125}}$

$\mathbb{E}[W_{(1)}^2] = (0)^2(0.66649375) + (1)^2(0.3185) + (2)^2(0.01500625) = 0.378525$

$\therefore \quad \sigma^2_{W_{(1)}} \equiv \mathbb{V}[W_{(1)}] = \mathbb{E}[W_{(1)}^2] - (\mathbb{E}[W_{(1)}])^2 = 0.378525 - (0.3485125)^2 = \boxed{\mathbf{0.257064037}}$