EX 5.4.1: Let random sample $X_1, X_2, \ldots, X_{20} \overset{iid}{\sim} \text{Normal}(\mu = 3, \sigma^2 = 10)$.

(a) What is the probability that the sample mean is at most four?

First, realize that the sample size is $n = 20$.

Since the population is a normal distribution, $\overline{X} \sim \text{Normal}(\mu, \sigma^2/n) = \text{Normal}(\mu_{\overline{X}} = 3, \sigma^2_{\overline{X}} = 1/2)$

$$\implies \mathbb{P}(\overline{X} \leq 4) = \mathbb{P}\left(\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} \leq \frac{4-3}{\sqrt{1/2}}\right) = \mathbb{P}(Z \leq 1.4142) = \Phi(1.4142) \overset{LOOKUP}{\approx} \boxed{\mathbf{0.92073}}$$

(b) What is the probability that the sample mean is between 2.5 and 3.8?

$$\begin{aligned} \mathbb{P}(2.5 \leq \overline{X} \leq 3.8) &= \mathbb{P}(\overline{X} \leq 3.8) - \mathbb{P}(\overline{X} \leq 2.5) = \mathbb{P}\left(\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} \leq \frac{3.8-3}{\sqrt{1/2}}\right) - \mathbb{P}\left(\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} \leq \frac{2.5-3}{\sqrt{1/2}}\right) \\ &= \mathbb{P}(Z \leq 1.1314) - \mathbb{P}(Z \leq -0.7071) = \Phi(1.1314) - \Phi(-0.7071) = \Phi(1.1314) - [1 - \Phi(0.7071)] \\ &\overset{LOOKUP}{=} 0.87076 - [1 - 0.76115] = \boxed{\mathbf{0.62191}} \end{aligned}$$

(c) What is the probability that the sample mean is within two standard deviations of the population mean?

First, realize that the expected value of the sample mean is the population mean: $\mu_{\overline{X}} = \mu$.

$$\begin{aligned} \mathbb{P}(|\overline{X} - \mu_{\overline{X}}| \leq 2\sigma_{\overline{X}}) &= \mathbb{P}\left(\left|\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}}\right| \leq 2\right) = \mathbb{P}(|Z| \leq 2) = \mathbb{P}(-2 \leq Z \leq 2) = \mathbb{P}(Z \leq 2) - \mathbb{P}(Z \leq -2) \\ &= \Phi(2) - \Phi(-2) = \Phi(2) - [1 - \Phi(2)] = 2 \cdot \Phi(2) - 1 \overset{LOOKUP}{=} (2)(0.97725) - 1 = \boxed{\mathbf{0.9545}} \end{aligned}$$

EX 5.4.2: Let $X_1, X_2, \ldots, X_{60}$ be a random sample of some non-normal population with mean $\mu = 10$ & variance $\sigma^2 = 80$.

(a) What is the approximate probability that the sample mean is at most eight?

First, realize that the sample size is $n = 60$.

Since $n > 30$, the Central Limit Theorem (CLT) asserts that $\overline{X} \overset{approx}{\sim} \text{Normal}(\mu, \sigma^2/n) = \text{Normal}(\mu_{\overline{X}} = 10, \sigma^2_{\overline{X}} = 4/3)$

$$\therefore \mathbb{P}(\overline{X} \leq 8) = \mathbb{P}\left(\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} \leq \frac{8-\mu_{\overline{X}}}{\sigma_{\overline{X}}}\right) \overset{CLT}{\approx} \mathbb{P}\left(Z \leq \frac{8-10}{\sqrt{4/3}}\right) = \Phi(-1.7321) = 1 - \Phi(1.7321) \overset{LOOKUP}{\approx} 1 - 0.95818 = \boxed{\mathbf{0.04182}}$$

(b) What is the approximate probability that the sample mean is at least 12.5?

$$\begin{aligned} \mathbb{P}(\overline{X} \geq 12.5) &= 1 - \mathbb{P}(\overline{X} \leq 12.5) = 1 - \mathbb{P}\left(\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} \leq \frac{12.5-\mu_{\overline{X}}}{\sigma_{\overline{X}}}\right) \overset{CLT}{\approx} 1 - \mathbb{P}\left(Z \leq \frac{12.5-10}{\sqrt{4/3}}\right) \\ &= 1 - \Phi(2.1651) \overset{LOOKUP}{\approx} 1 - 0.98500 = \boxed{\mathbf{0.015}} \end{aligned}$$

(c) What is the approximate probability that the sample mean is outside 1.5 standard deviations of the population mean?

First, realize that the expected value of the sample mean is the population mean: $\mu_{\overline{X}} = \mu$.

$$\begin{aligned} \mathbb{P}(|\overline{X} - \mu_{\overline{X}}| > 1.5\sigma_{\overline{X}}) &= \mathbb{P}\left(\left|\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}}\right| > 1.5\right) \\ &\overset{CLT}{\approx} \mathbb{P}(|Z| > 1.5) \\ &= \mathbb{P}(Z < -1.5 \text{ or } Z > 1.5) \\ &= \mathbb{P}(Z < -1.5 \ \cup \ Z > 1.5) \\ &\overset{PD}{=} \mathbb{P}(Z < -1.5) + \mathbb{P}(Z > 1.5) \\ &= \mathbb{P}(Z \leq -1.5) + [1 - \mathbb{P}(Z \leq 1.5)] \\ &= \Phi(-1.5) + [1 - \Phi(1.5)] \\ &= [1 - \Phi(1.5)] + [1 - \Phi(1.5)] \\ &= 2 \cdot [1 - \Phi(1.5)] \\ &\overset{LOOKUP}{\approx} 2 \cdot [1 - 0.93319] = \boxed{\mathbf{0.13362}} \end{aligned}$$

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EX 5.4.3: Suppose there are fifty plane flights from the Dallas airport to the Lubbock airport in a single day. Moreover, suppose that for every ten flights on a given day, three of them arrive late to the gate.

(a) What is the approximate probability that exactly ten flights are late in a day?

Let $X \equiv$ (# late flights in a day). Then "Success" $\equiv$ "Flight is late" $\implies \begin{matrix} p \equiv \mathbb{P}(\text{Success}) = 3/10 = 0.3 \\ q = 1 - p = 1 - 0.3 = 0.7 \end{matrix}$

$\therefore\ X \sim \text{Binomial}(n = 50, p = 0.3)$. Since $\begin{matrix} np = (50)(0.3) = 15 \geq 10 \\ nq = (50)(0.7) = 35 \geq 10 \end{matrix}$, $X \overset{approx}{\sim} \text{Normal}(\mu_X = np, \sigma_X^2 = npq)$

$$\begin{aligned} \mathbb{P}(X = 10) &= \mathbb{P}(X \leq 10) - \mathbb{P}(X \leq 9) = \text{Bi}(10; n, p) - \text{Bi}(9; n, p) \\ &\overset{CLT}{\approx} \Phi\left(\frac{10 + 0.5 - np}{\sqrt{npq}}\right) - \Phi\left(\frac{9 + 0.5 - np}{\sqrt{npq}}\right) = \Phi\left(\frac{10.5 - (50)(0.3)}{\sqrt{(50)(0.3)(0.7)}}\right) - \Phi\left(\frac{9.5 - (50)(0.3)}{\sqrt{(50)(0.3)(0.7)}}\right) \\ &= \Phi(-1.3887) - \Phi(-1.6973) = [1 - \Phi(1.3887)] - [1 - \Phi(1.6973)] = \Phi(1.6973) - \Phi(1.3887) \\ &\overset{LOOKUP}{\approx} 0.95543 - 0.91774 = \boxed{\mathbf{0.03769}} \end{aligned}$$

(b) What is the approximate probability that at most ten flights are late in a day?

$$\mathbb{P}(X \leq 10) = \text{Bi}(10; n, p) \overset{CLT}{\approx} \Phi\left(\frac{10 + 0.5 - np}{\sqrt{npq}}\right) = \Phi(-1.3887) = 1 - \Phi(1.3887) \overset{LOOKUP}{\approx} 1 - 0.91774 = \boxed{\mathbf{0.08226}}$$

(c) What is the approximate probability that at least ten flights are late in a day?

$$\begin{aligned} \mathbb{P}(X \geq 10) &= 1 - \mathbb{P}(X < 10) = 1 - \mathbb{P}(X \leq 9) = 1 - \text{Bi}(9; n, p) \overset{CLT}{\approx} 1 - \Phi\left(\frac{9 + 0.5 - np}{\sqrt{npq}}\right) = 1 - \Phi\left(\frac{9.5 - (50)(0.3)}{\sqrt{(50)(0.3)(0.7)}}\right) \\ &= 1 - \Phi(-1.6973) = 1 - [1 - \Phi(1.6973)] = \Phi(1.6973) \overset{LOOKUP}{\approx} \boxed{\mathbf{0.95543}} \end{aligned}$$

EX 5.4.4: A study of a bank's teller lines indicated that all day four customers are expected to be waiting per hour. Assume that the number of customers waiting in line is modeled by a Poisson distribution.

(a) What is the approximate probability that upon visiting the bank from 9am to 7pm that thirty customers are waiting?

Let $X \equiv$ (# Customers waiting in line from 9am to 7pm at bank) Then, $X \sim \text{Poisson}(\lambda = \underbrace{4}_{\alpha} \cdot \underbrace{10}_{\Delta t}) = \text{Poisson}(\lambda = 40)$

$\therefore\ X \sim \text{Poisson}(\lambda = 40)$. Since $\lambda > 20$, $X \overset{approx}{\sim} \text{Normal}(\mu_X = \lambda, \sigma_X^2 = \lambda)$

$$\begin{aligned} \mathbb{P}(X = 30) &= \mathbb{P}(X \leq 30) - \mathbb{P}(X \leq 29) = \text{Pois}(30; \lambda) - \text{Pois}(29; \lambda) \\ &\overset{CLT}{\approx} \Phi\left(\frac{30 + 0.5 - \lambda}{\sqrt{\lambda}}\right) - \Phi\left(\frac{29 + 0.5 - \lambda}{\sqrt{\lambda}}\right) = \Phi\left(\frac{30.5 - 40}{\sqrt{40}}\right) - \Phi\left(\frac{29.5 - 40}{\sqrt{40}}\right) \\ &= \Phi(-1.5021) - \Phi(-1.6602) = [1 - \Phi(1.5021)] - [1 - \Phi(1.6602)] = \Phi(1.6602) - \Phi(1.5021) \\ &\overset{LOOKUP}{\approx} 0.95154 - 0.93319 = \boxed{\mathbf{0.01835}} \end{aligned}$$

(b) What is the approx. probability that upon visiting the bank from 9am to 7pm that at least 30 customers are waiting?

$$\begin{aligned} \mathbb{P}(X \geq 30) &= 1 - \mathbb{P}(X < 30) = 1 - \mathbb{P}(X \leq 29) = 1 - \text{Pois}(29; \lambda) \\ &\overset{CLT}{\approx} 1 - \Phi\left(\frac{29 + 0.5 - \lambda}{\sqrt{\lambda}}\right) = 1 - \Phi\left(\frac{29.5 - 40}{\sqrt{40}}\right) \\ &= 1 - \Phi(-1.6602) = 1 - [1 - \Phi(1.6602)] = \Phi(1.6602) \\ &\overset{LOOKUP}{\approx} \boxed{\mathbf{0.95154}} \end{aligned}$$

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