EX 6.1.1: Given the following sample from a hypothetical population of all fuel efficiencies of 6-cylinder vehicles (in mpg):

$$x: \quad 21.0,\ 15.0,\ 21.0,\ 21.4,\ 18.1,\ 19.2,\ 17.8,\ 19.7,\ 13.0,\ 35.0$$

(a) Compute the point estimate of the population mean fuel efficiency μ.

Compute the sample mean since it's an unbiased estimator of the population mean.

$$\overline{x} = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{10}(21.0+15.0+21.0+21.4+18.1+19.2+17.8+19.7+13.0+35.0) = \left(\frac{1}{10}\right)(201.2) = \boxed{20.12 \text{ mpg}}$$

(b) Compute the point estimate of the population variance of the fuel efficiency σ^2.

Compute the sample variance since it's an unbiased estimator of the population variance.

$$S_{xx} = \sum_{k=1}^{n} x_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} x_k\right)^2 = 4360.14 - \left(\frac{1}{10}\right)(201.2)^2 = 311.996 \implies s^2 = \frac{S_{xx}}{n-1} = \frac{311.996}{9} \approx \boxed{34.6662 \text{ (mpg)}^2}$$

(c) Compute the point estimate of the proportion of all such 6-cylinder vehicles whose fuel efficiency is less than 20 mpg.

$$\widehat{p} = \frac{X}{n} = \frac{(\#\text{ data points in sample } < 20 \text{ mpg})}{(\text{Sample Size})} = \frac{6}{10} = \boxed{0.60} \qquad \widehat{q} = 1 - \widehat{p} = 1 - 0.60 = 0.40$$

Note that in this context, $X \sim \text{Binomial}(n, \widehat{p}) \implies \mathbb{E}[X] = n\widehat{p}$ and $\mathbb{V}[X] = n\widehat{p}\widehat{q}$

(d) Compute the estimated standard error $\widehat{\sigma}_{\overline{X}}$ of the point estimator $\overline{X}$.

$$\widehat{\sigma}_{\overline{X}} = \sqrt{\mathbb{V}[\overline{X}]} = \sqrt{\widehat{\sigma}^2/n} = \sqrt{s^2/n} \approx \sqrt{34.6662/10} = \sqrt{3.46662} \approx \boxed{1.8619}$$

(e) Compute the estimated standard error $\widehat{\sigma}_{\widehat{p}}$ of the point estimator $\widehat{p} := X/n$.

$$\widehat{\sigma}_{\widehat{p}} = \sqrt{\mathbb{V}[X/n]} = \sqrt{\frac{1}{n^2}\mathbb{V}[X]} = \frac{1}{n}\sqrt{\mathbb{V}[X]} = \frac{1}{n}\sqrt{n\widehat{p}\widehat{q}} = \frac{1}{10}\sqrt{(10)(0.6)(0.4)} \approx \boxed{0.1549}$$

EX 6.1.2: Given a random sample $X_1, \ldots, X_{n_1}$ from a population with mean μ_1 and variance σ_1^2, and given a random sample $Y_1, \ldots, Y_{n_2}$ from a population with mean μ_2 and variance σ_2^2.

(a) Show that the point estimator $\overline{X} - \overline{Y}$ is an unbiased estimator of $\mu_1 - \mu_2$.

$$\mathbb{E}[\overline{X} - \overline{Y}] = \mathbb{E}[\overline{X}] - \mathbb{E}[\overline{Y}] = \mu_1 - \mu_2$$

(b) Find the expression for the standard error of the point estimator $\overline{X} - \overline{Y}$.

$$\sigma_{\overline{X}-\overline{Y}} = \sqrt{\mathbb{V}[\overline{X} - \overline{Y}]} = \sqrt{\mathbb{V}[\overline{X}] + \mathbb{V}[\overline{Y}]} = \boxed{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

(c) Find the expression for the estimated standard error of the point estimator $\overline{X} - \overline{Y}$.

$$\widehat{\sigma}_{\overline{X}-\overline{Y}} = \sqrt{\frac{\widehat{\sigma}_1^2}{n_1} + \frac{\widehat{\sigma}_2^2}{n_2}} = \boxed{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

where s_1^2 is the sample variance of $x_1, \ldots, x_{n_1}$
s_2^2 is the sample variance of $y_1, \ldots, y_{n_2}$

EX 6.1.3: Let random sample $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$.

(a) Find the expression for the standard error of the sample mean $\overline{X}$.

First, realize that $\begin{array}{l} X_1 \sim \text{Poisson}(\lambda) \implies \mathbb{E}[X_1] = \lambda, \quad \mathbb{V}[X_1] = \lambda \\ X_2 \sim \text{Poisson}(\lambda) \implies \mathbb{E}[X_2] = \lambda, \quad \mathbb{V}[X_2] = \lambda \end{array}$

$$\sigma_{\overline{X}} = \sqrt{\mathbb{V}[\overline{X}]} = \sqrt{\mathbb{V}\left[\tfrac{1}{2}(X_1 + X_2)\right]} = \sqrt{\tfrac{1}{4}(\mathbb{V}[X_1] + \mathbb{V}[X_2])} = \tfrac{1}{2}\sqrt{\lambda + \lambda} = \tfrac{1}{2}\sqrt{2\lambda} = \boxed{\sqrt{\lambda/2}}$$

(b) Show that point estimator $\dfrac{5X_1 - 2X_2}{3}$ is an unbiased estimator of the parameter λ.

$$\mathbb{E}\left[\frac{5X_1 - 2X_2}{3}\right] = \mathbb{E}\left[\frac{5}{3}X_1 - \frac{2}{3}X_2\right] = \frac{5}{3}\cdot\mathbb{E}[X_1] - \frac{2}{3}\cdot\mathbb{E}[X_2] = \frac{5}{3}\lambda - \frac{2}{3}\lambda = \lambda$$

$\therefore \ \mathbb{E}\left[\dfrac{5X_1 - 2X_2}{3}\right] = \lambda \implies$ point estimator $\dfrac{5X_1 - 2X_2}{3}$ <u>is</u> an unbiased estimator of λ

(c) Show that point estimator $\dfrac{7X_1 + 3X_2}{10}$ is an unbiased estimator of the parameter λ.

$$\mathbb{E}\left[\frac{7X_1 + 3X_2}{10}\right] = \mathbb{E}\left[\frac{7}{10}X_1 + \frac{3}{10}X_2\right] = \frac{7}{10}\cdot\mathbb{E}[X_1] + \frac{3}{10}\cdot\mathbb{E}[X_2] = \frac{7}{10}\lambda + \frac{3}{10}\lambda = \lambda$$

$\therefore \ \mathbb{E}\left[\dfrac{7X_1 + 3X_2}{10}\right] = \lambda \implies$ point estimator $\dfrac{7X_1 + 3X_2}{10}$ <u>is</u> an unbiased estimator of λ

(d) Which of the two point estimators, $\dfrac{5X_1 - 2X_2}{3}$ and $\dfrac{7X_1 + 3X_2}{10}$, is a better estimator of λ?

$$\mathbb{V}\left[\frac{5X_1 - 2X_2}{3}\right] = \mathbb{V}\left[\frac{5}{3}X_1 - \frac{2}{3}X_2\right] = \left(\frac{5}{3}\right)^2\cdot\mathbb{V}[X_1] + \left(\frac{2}{3}\right)^2\cdot\mathbb{V}[X_2] = \frac{25}{9}\lambda + \frac{4}{9}\lambda = \frac{29}{9}\lambda$$

$$\mathbb{V}\left[\frac{7X_1 + 3X_2}{10}\right] = \mathbb{V}\left[\frac{7}{10}X_1 + \frac{3}{10}X_2\right] = \left(\frac{7}{10}\right)^2\cdot\mathbb{V}[X_1] + \left(\frac{3}{10}\right)^2\cdot\mathbb{V}[X_2] = \frac{49}{100}\lambda + \frac{9}{100}\lambda = \frac{58}{100}\lambda$$

Since $\lambda > 0$, $\dfrac{58}{100}\lambda < \dfrac{29}{9}\lambda \implies \mathbb{V}\left[\dfrac{7X_1 + 3X_2}{10}\right] < \mathbb{V}\left[\dfrac{5X_1 - 2X_2}{3}\right] \implies \boxed{\dfrac{7X_1 + 3X_2}{10} \text{ is a better estimator of } \lambda}$

(e) Show that point estimator $X_1^2 - X_2$ is <u>not</u> an unbiased estimator of the parameter λ.

$$\mathbb{E}[X_1^2 - X_2] = \mathbb{E}[X_1^2] - \mathbb{E}[X_2] = \left[\mathbb{V}[X_1] + (\mathbb{E}[X_1])^2\right] - \mathbb{E}[X_2] = [\lambda + \lambda^2] - \lambda = \lambda^2$$

$\therefore \ \mathbb{E}[X_1^2 - X_2] \neq \lambda \implies$ point estimator $X_1^2 - X_2$ is <u>not</u> an unbiased estimator of λ

EX 6.1.4: Sometimes it's easier to find a point estimator of a <u>function</u> of a population parameter.

Let random sample $X_1, \ldots, X_n \overset{iid}{\sim}$ pdf $f_X(x;\theta) := \dfrac{\theta}{x^4}$ for $0 < \theta \le x < \infty$

(a) Show that the sample mean $\overline{X}$ is <u>not</u> an unbiased estimator of $1/\theta$.

$$\mathbb{E}[\overline{X}] = \mu = \mathbb{E}[X_1] = \int_{\text{Supp}(X_1)} x\cdot f_X(x;\theta)\ dx = \int_\theta^\infty x\cdot\frac{\theta}{x^4}\ dx = \int_\theta^\infty \theta x^{-3}\ dx$$

$$= \left[\frac{\theta x^{-3+1}}{-3+1}\right]_{x=\theta}^{x\to\infty} = \left[-\frac{\theta}{2x^2}\right]_{x=\theta}^{x\to\infty} \overset{FTC}{=} 0 - \left(-\frac{\theta}{2\theta^2}\right) = \frac{1}{2\theta}$$

$\therefore \ \mathbb{E}[\overline{X}] \neq 1/\theta \implies$ sample mean $\overline{X}$ is <u>not</u> an unbiased estimator of $1/\theta$

(b) Based on your work from the previous part, construct an unbiased estimator of $1/\theta$.

$\mathbb{E}[\overline{X}] = \dfrac{1}{2\theta}$ suggests that $\boxed{2\overline{X}}$ <u>is</u> an unbiased estimator of $1/\theta$

$$\mathbb{E}[2\overline{X}] = 2\cdot\mathbb{E}[\overline{X}] = (2)\left(\frac{1}{2\theta}\right) = \frac{1}{\theta}$$