POINT ESTIMATION: UNBIASED ESTIMATORS [DEVORE 6.1]

- **POINT ESTIMATORS:** Given a random sample $X_1, \dots, X_n$ from a population with parameter θ.

  Then a **point estimator** $\widehat{\theta}$ of parameter θ is a suitable statistic T of sample: $\widehat{\theta} = T(X_1, \dots, X_n)$

  Recall from Chapter 5 that a statistic T is a function of the random sample.

  NOTE: Often there are several point estimators for a population parameter.

POPULATION PARAMETER	POINT ESTIMATOR(S)
Mean μ	$\widehat{\mu} := \overline{X},\ \widehat{\mu} := \widetilde{X},\ \widehat{\mu} := \overline{X}_{tr(10\%)}$
Proportion p	$\widehat{p} := X/n$
Variance σ^2	$\widehat{\sigma}^2 := S^2$

Random sample $X_1, \dots, X_{n_1}$ from population with mean μ_1 & variance σ_1^2

Random sample $Y_1, \dots, Y_{n_2}$ from population with mean μ_2 & variance σ_2^2

POPULATION PARAMETER	POINT ESTIMATOR(S)
Mean Difference $\mu_1 - \mu_2$	$\widehat{\mu}_1 - \widehat{\mu}_2 := \overline{X} - \overline{Y}$
Proportion Difference $p_1 - p_2$	$\widehat{p}_1 - \widehat{p}_2 := X/n_1 - Y/n_2$
Variance Ratio σ_1^2/σ_2^2	$\widehat{\sigma}_1^2/\widehat{\sigma}_2^2 := \frac{(n_1-1)S_1^2}{(n_2-1)S_2^2}$

- **UNBIASED POINT ESTIMATORS:** Given a population with parameter θ.

  Then a point estimator $\widehat{\theta}$ is an **unbiased estimator** of θ if $\mathbb{E}[\ \widehat{\theta}\ ] = \theta$

  Otherwise, the point estimator is **biased** with a **bias** of $\text{Bias}[\ \widehat{\theta}\ ] := \mathbb{E}[\ \widehat{\theta}\ ] - \theta$

  i.e. A point estimator is unbiased if its sampling distribution is always "centered" at true value of population parameter.

  If $\text{Bias}[\ \widehat{\theta}\ ] < 0$, then $\widehat{\theta}$ tends to underestimate the population parameter value.

  If $\text{Bias}[\ \widehat{\theta}\ ] > 0$, then $\widehat{\theta}$ tends to overestimate the population parameter value.

- **UNBIASED ESTIMATORS OF POPULATION MEAN & POPULATION VARIANCE:**

  Given a random sample $X_1, \dots, X_n$ from a population with mean μ and variance σ^2.

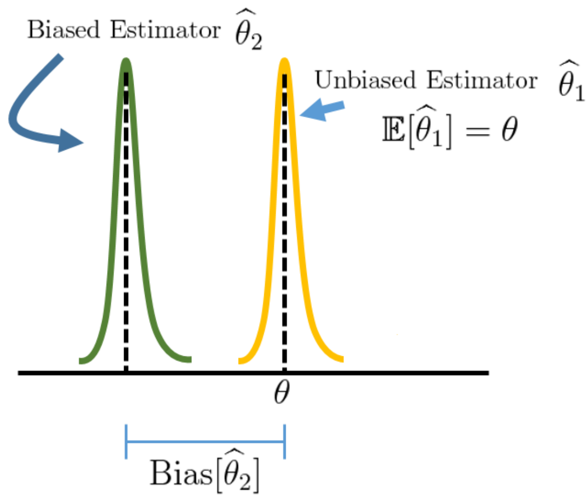
  Then $\overline{X}$ and S^2 are unbiased estimators of μ and σ^2, respectively.

- **UNBIASED ESTIMATOR OF POPULATION PROPORTION:**

  Let random variable $X \sim \text{Binomial}(n, p)$.

  Then sample proportion X/n is an unbiased estimator of pop. proportion p.

The two curves are the pdf's of the sampling distributions of $\widehat{\theta}_1$ and $\widehat{\theta}_2$



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POINT ESTIMATION: UMVUE'S, STANDARD ERRORS [DEVORE 6.1]

- **UMVUE:** The "best" unbiased estimator is the one that varies the least:

  The unbiased estimator with smallest variance is the **uniformly minimum variance unbiased estimator (UMVUE)**.

- **UMVUE OF NORMAL POPULATION:**

  Let random sample $X_1, \ldots, X_n \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$.

  Then $\overline{X}$ is the UMVUE for μ.

- **STANDARD ERROR:** The **standard error** of a point estimator $\widehat{\theta}$ is $\sigma_{\widehat{\theta}} := \sqrt{\mathbb{V}[\,\widehat{\theta}\,]}$

- **ESTIMATED STANDARD ERROR:** The **estimated standard error** of a point estimator, denoted $\widehat{\sigma}_{\widehat{\theta}}$, is the value of the standard error $\sigma_{\widehat{\theta}}$ when any unknown parameters involved in its expression have themselves been estimated.

  Standard Error of $\overline{X}$ is $\sigma_{\overline{X}} = \sigma/\sqrt{n}$

  Estimated Standard Error of $\overline{X}$ is $\widehat{\sigma}_{\overline{X}} = \widehat{\sigma}/\sqrt{n} = s/\sqrt{n}$

- **EXPECTATION & VARIANCE (REVISITED):**

  Let $X_1, \ldots, X_n$ be a random sample from a population and $c_1, \ldots, c_n \neq 0$. Then:

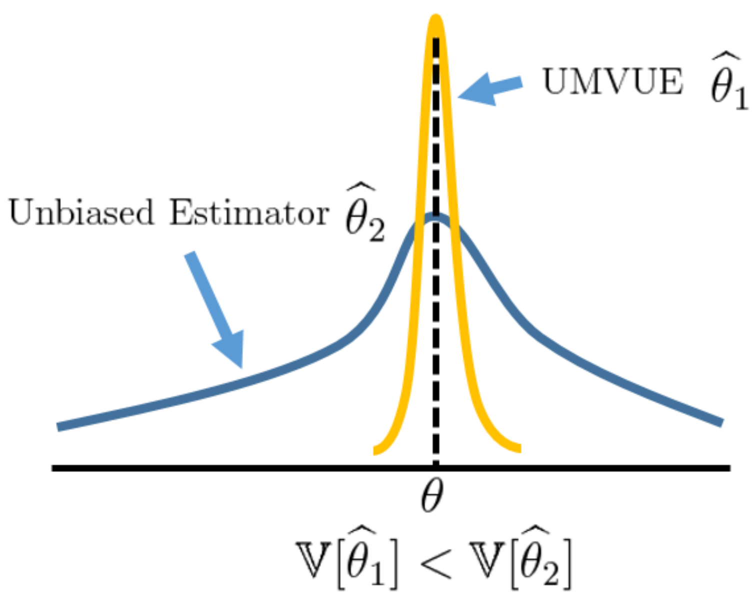
  (i) $\mathbb{E}[c_1 X_1 + \cdots + c_n X_n] = c_1 \mathbb{E}[X_1] + \cdots + c_n \mathbb{E}[X_n]$

  (ii) $\mathbb{V}[c_1 X_1 + \cdots + c_n X_n] = c_1^2 \mathbb{V}[X_1] + \cdots + c_n^2 \mathbb{V}[X_n]$

  (iii) $\mathbb{E}[X_1 - X_2] = \mathbb{E}[X_1] - \mathbb{E}[X_2]$

  (iv) $\mathbb{V}[X_1 - X_2] = \mathbb{V}[X_1] + \mathbb{V}[X_2]$

The two curves are the pdf's of the sampling distributions of $\widehat{\theta}_1$ and $\widehat{\theta}_2$



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EX 6.1.1: Given the following sample from a hypothetical population of all fuel efficiencies of 6-cylinder vehicles (in mpg):

$$x: \quad 21.0,\ 15.0,\ 21.0,\ 21.4,\ 18.1,\ 19.2,\ 17.8,\ 19.7,\ 13.0,\ 35.0$$

(a) Compute the point estimate of the population mean fuel efficiency μ.

(b) Compute the point estimate of the population variance of the fuel efficiency σ^2.

(c) Compute the point estimate of the proportion of all such 6-cylinder vehicles whose fuel efficiency is less than 20 mpg.

(d) Compute the estimated standard error $\widehat{\sigma}_{\overline{X}}$ of the point estimator $\overline{X}$.

(e) Compute the estimated standard error $\widehat{\sigma}_{\widehat{p}}$ of the point estimator $\widehat{p} := X/n$.

EX 6.1.2: Given a random sample $X_1, \ldots, X_{n_1}$ from a population with mean μ_1 and variance σ_1^2, and given a random sample $Y_1, \ldots, Y_{n_2}$ from a population with mean μ_2 and variance σ_2^2.

(a) Show that the point estimator $\overline{X} - \overline{Y}$ is an unbiased estimator of $\mu_1 - \mu_2$.

(b) Find the expression for the standard error of the point estimator $\overline{X} - \overline{Y}$.

(c) Find the expression for the estimated standard error of the point estimator $\overline{X} - \overline{Y}$.

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EX 6.1.3: Let random sample $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$.

(a) Find the expression for the standard error of the sample mean $\overline{X}$.

(b) Show that point estimator $\dfrac{5X_1 - 2X_2}{3}$ is an unbiased estimator of the parameter λ.

(c) Show that point estimator $\dfrac{7X_1 + 3X_2}{10}$ is an unbiased estimator of the parameter λ.

(d) Which of the two point estimators, $\dfrac{5X_1 - 2X_2}{3}$ and $\dfrac{7X_1 + 3X_2}{10}$, is a better estimator of λ?

(e) Show that point estimator $X_1^2 - X_2$ is <u>not</u> an unbiased estimator of the parameter λ.

EX 6.1.4: Sometimes it's easier to find a point estimator of a <u>function</u> of a population parameter.

Let random sample $X_1, \ldots, X_n \overset{iid}{\sim}$ pdf $f_X(x;\theta) := \dfrac{\theta}{x^4}$ for $0 < \theta \le x < \infty$

(a) Show that the sample mean $\overline{X}$ is <u>not</u> an unbiased estimator of $1/\theta$.

(b) Based on your work from the previous part, construct an unbiased estimator of $1/\theta$.

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