HELMERT'S χ^2 DISTRIBUTION [DEVORE 7.4]

- **HELMERT'S χ^2 DISTRIBUTION (SUMMARY):**

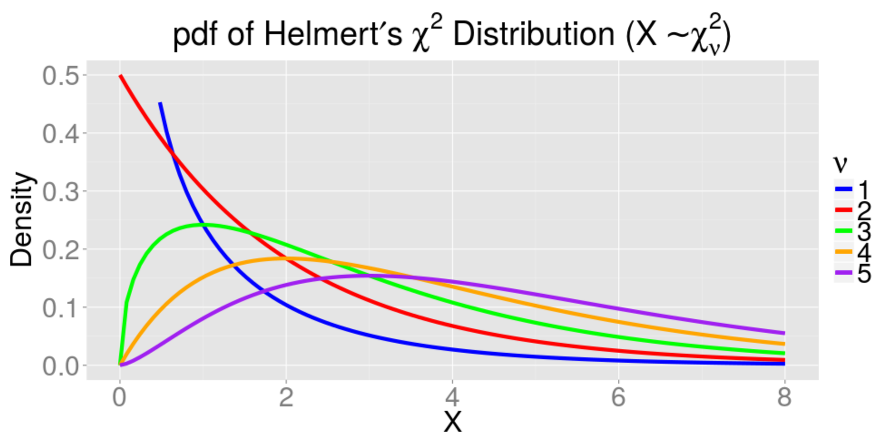
Notation	$X \sim \chi^2_\nu$
Parameter(s)	$\nu \equiv \#$ Degrees of Freedom $(\nu = 1, 2, 3, 4, \cdots)$
Support	$\text{Supp}(X) = (0, \infty)$
pdf	$f_X(x; \nu) := \frac{1}{2^{\nu/2} \cdot \Gamma(\nu/2)} \cdot x^{(\nu/2)-1} e^{-x/2}$
cdf	$\Phi_{\chi^2}(x; \nu) = \frac{1}{2^{\nu/2} \cdot \Gamma(\nu/2)} \int_0^x \xi^{(\nu/2)-1} e^{-\xi/2} \, d\xi$
Mean	$\mathbb{E}[X] = \nu$
Variance	$\mathbb{V}[X] = 2\nu$
Model(s)	(Used exclusively for Statistical Inference)

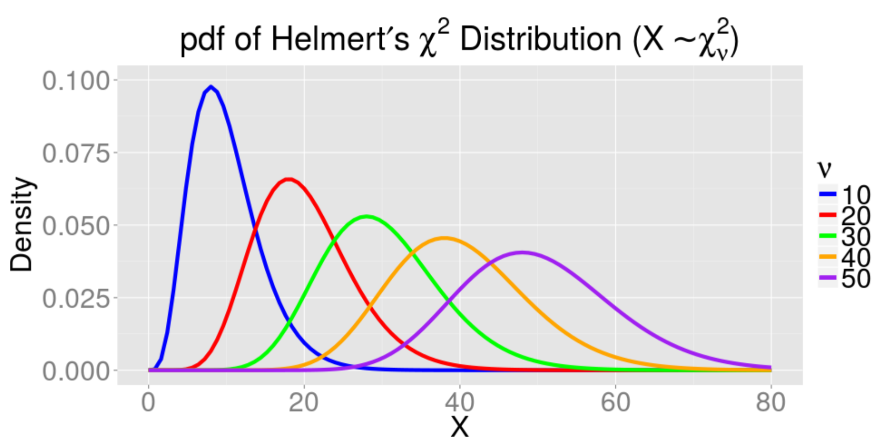
χ is the lowercase Greek letter "chi" (pronounced KEYE)

NOTE: χ^2 distributions are special cases of Gamma distributions.

- **HELMERT'S χ^2 DISTRIBUTION (PROPERTIES):**
  - The χ^2_ν pdf curve is positively skewed.
  - The χ^2_ν pdf curve becomes more symmetric as ν increases.
  - For $\nu > 40$, the χ^2_ν pdf curve is very close to Normal$(\mu = \nu, \sigma^2 = 2\nu)$ pdf.

- **HELMERT'S χ^2 DISTRIBUTION (PLOTS):**





1-SAMPLE INFERENCE: SMALL-SAMPLE CI'S FOR VARIANCE [DEVORE 7.4]

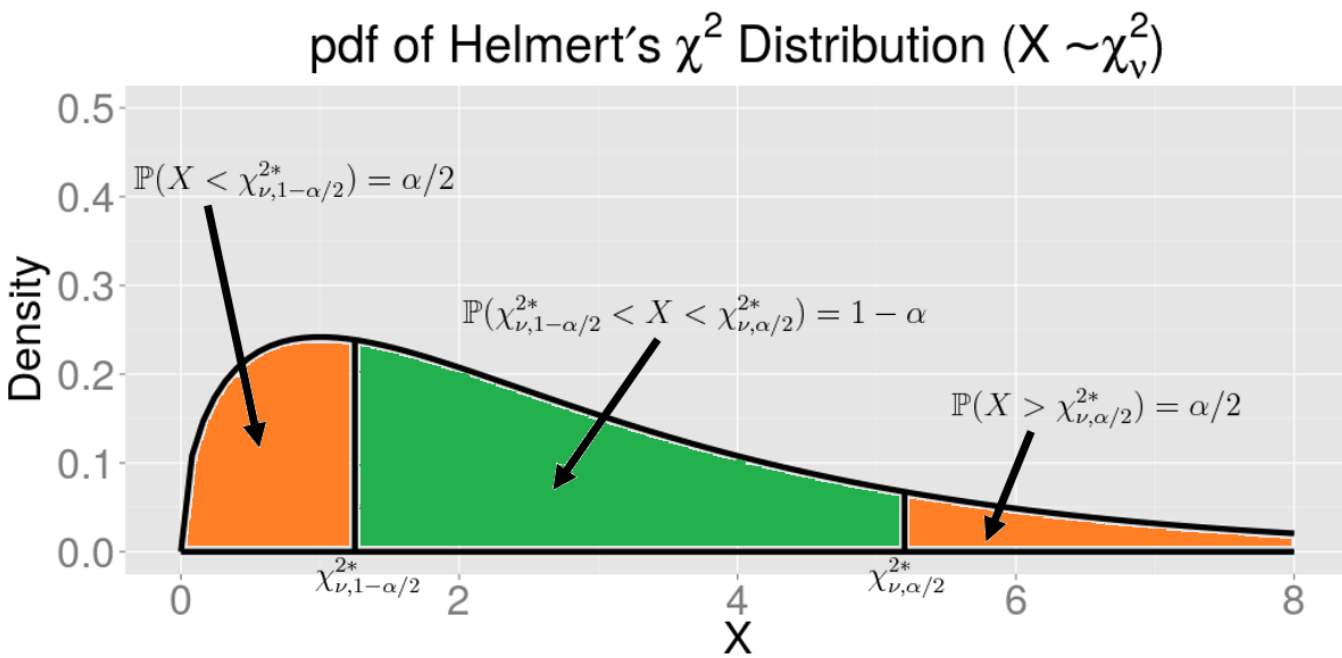
- **χ^2 CRITICAL VALUES (DEFN):** $\chi^{2*}_{\nu,\alpha/2}$ is called a χ^2 **critical value** of the χ^2 distribution with ν df's such that its upper-tail probability is exactly its subscript value $\alpha/2$: (Here, $X \sim \chi^2_\nu$)

$$\mathbb{P}(X < \chi^{2*}_{\nu,1-\alpha/2}) = \alpha/2 \qquad \mathbb{P}(X > \chi^{2*}_{\nu,\alpha/2}) = \alpha/2$$

IMPORTANT: Do not confuse χ^2 critical values with χ^2 percentiles:

$$\mathbb{P}(X \le \chi^{2}_{\nu,1-\alpha/2}) = 1-\alpha/2 \qquad \mathbb{P}(X \le \chi^{2}_{\nu,\alpha/2}) = \alpha/2$$

Finally, notice that $\chi^{2*}_{\nu,\alpha/2}$ and $\chi^{2*}_{\nu,1-\alpha/2}$ are always **positive**.



- **A STATISTIC RELATED TO χ^2 DISTRIBUTION:** Let $X_1, \ldots, X_n \overset{iid}{\sim}$ Normal(μ, σ^2) population. Then:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

- **SMALL-SAMPLE CI FOR NORMAL POPULATION VARIANCE σ^2 :**

Given a normal population with unknown mean μ and variance σ^2.

Let $x_1, \ldots, x_n$ be a small sample taken from the population.

Then the $100(1-\alpha)\%$ **small-sample CI for** σ^2 is $\left(\dfrac{(n-1)s^2}{\chi^{2*}_{n-1,\alpha/2}}, \dfrac{(n-1)s^2}{\chi^{2*}_{n-1,1-\alpha/2}}\right)$

- **SMALL-SAMPLE CI FOR NORMAL POPULATION STANDARD DEVIATION σ :**

Given a normal population with unknown mean μ and variance σ^2.

Let $x_1, \ldots, x_n$ be a small sample taken from the population.

Then the $100(1-\alpha)\%$ **small-sample CI for** σ is $\left(\sqrt{\dfrac{(n-1)s^2}{\chi^{2*}_{n-1,\alpha/2}}}, \sqrt{\dfrac{(n-1)s^2}{\chi^{2*}_{n-1,1-\alpha/2}}}\right)$

HELMERT'S χ^2 CRITICAL VALUES

$$\mathbb{P}(\chi^{2*}_{\nu,1-\alpha/2} < X < \chi^{2*}_{\nu,\alpha/2}) = 1-\alpha$$

ν	**80% CI** ($\alpha/2 = 0.1$)		**90% CI** ($\alpha/2 = 0.05$)		**95% CI** ($\alpha/2 = 0.025$)		**98% CI** ($\alpha/2 = 0.01$)		**99% CI** ($\alpha/2 = 0.005$)	
	$\chi^{2*}_{\nu,1-\alpha/2}$	$\chi^{2*}_{\nu,\alpha/2}$	$\chi^{2*}_{\nu,1-\alpha/2}$	$\chi^{2*}_{\nu,\alpha/2}$	$\chi^{2*}_{\nu,1-\alpha/2}$	$\chi^{2*}_{\nu,\alpha/2}$	$\chi^{2*}_{\nu,1-\alpha/2}$	$\chi^{2*}_{\nu,\alpha/2}$	$\chi^{2*}_{\nu,1-\alpha/2}$	$\chi^{2*}_{\nu,\alpha/2}$
1	0.016	2.706	0.004	3.841	0.001	5.024	0.000	6.635	0.000	7.879
2	0.211	4.605	0.103	5.991	0.051	7.378	0.020	9.210	0.010	10.597
3	0.584	6.251	0.352	7.815	0.216	9.348	0.115	11.345	0.072	12.838
4	1.064	7.779	0.711	9.488	0.484	11.143	0.297	13.277	0.207	14.860
5	1.610	9.236	1.145	11.070	0.831	12.833	0.554	15.086	0.412	16.750
6	2.204	10.645	1.635	12.592	1.237	14.449	0.872	16.812	0.676	18.548
7	2.833	12.017	2.167	14.067	1.690	16.013	1.239	18.475	0.989	20.278
8	3.490	13.362	2.733	15.507	2.180	17.535	1.646	20.090	1.344	21.955
9	4.168	14.684	3.325	16.919	2.700	19.023	2.088	21.666	1.735	23.589
10	4.865	15.987	3.940	18.307	3.247	20.483	2.558	23.209	2.156	25.188
11	5.578	17.275	4.575	19.675	3.816	21.920	3.053	24.725	2.603	26.757
12	6.304	18.549	5.226	21.026	4.404	23.337	3.571	26.217	3.074	28.300
13	7.042	19.812	5.892	22.362	5.009	24.736	4.107	27.688	3.565	29.819
14	7.790	21.064	6.571	23.685	5.629	26.119	4.660	29.141	4.075	31.319
15	8.547	22.307	7.261	24.996	6.262	27.488	5.229	30.578	4.601	32.801
16	9.312	23.542	7.962	26.296	6.908	28.845	5.812	32.000	5.142	34.267
17	10.085	24.769	8.672	27.587	7.564	30.191	6.408	33.409	5.697	35.718
18	10.865	25.989	9.390	28.869	8.231	31.526	7.015	34.805	6.265	37.156
19	11.651	27.204	10.117	30.144	8.907	32.852	7.633	36.191	6.844	38.582
20	12.443	28.412	10.851	31.410	9.591	34.170	8.260	37.566	7.434	39.997
21	13.240	29.615	11.591	32.671	10.283	35.479	8.897	38.932	8.034	41.401
22	14.041	30.813	12.338	33.924	10.982	36.781	9.542	40.289	8.643	42.796
23	14.848	32.007	13.091	35.172	11.689	38.076	10.196	41.638	9.260	44.181
24	15.659	33.196	13.848	36.415	12.401	39.364	10.856	42.980	9.886	45.559
25	16.473	34.382	14.611	37.652	13.120	40.646	11.524	44.314	10.520	46.928
26	17.292	35.563	15.379	38.885	13.844	41.923	12.198	45.642	11.160	48.290
27	18.114	36.741	16.151	40.113	14.573	43.195	12.879	46.963	11.808	49.645
28	18.939	37.916	16.928	41.337	15.308	44.461	13.565	48.278	12.461	50.993
29	19.768	39.087	17.708	42.557	16.047	45.722	14.256	49.588	13.121	52.336
30	20.599	40.256	18.493	43.773	16.791	46.979	14.953	50.892	13.787	53.672
31	21.434	41.422	19.281	44.985	17.539	48.232	15.655	52.191	14.458	55.003
32	22.271	42.585	20.072	46.194	18.291	49.480	16.362	53.486	15.134	56.328
33	23.110	43.745	20.867	47.400	19.047	50.725	17.074	54.776	15.815	57.648
34	23.952	44.903	21.664	48.602	19.806	51.966	17.789	56.061	16.501	58.964
35	24.797	46.059	22.465	49.802	20.569	53.203	18.509	57.342	17.192	60.275
36	25.643	47.212	23.269	50.998	21.336	54.437	19.233	58.619	17.887	61.581
37	26.492	48.363	24.075	52.192	22.106	55.668	19.960	59.893	18.586	62.883
38	27.343	49.513	24.884	53.384	22.878	56.896	20.691	61.162	19.289	64.181
39	28.196	50.660	25.695	54.572	23.654	58.120	21.426	62.428	19.996	65.476
40	29.051	51.805	26.509	55.758	24.433	59.342	22.164	63.691	20.707	66.766

EX 7.4.1: Consider the normal population of all college student heights and the height variance σ^2 (in feet^2.)

(a) Suppose a sample of size $(n = 5)$ is taken from the population.

Moreover, the sample mean $\overline{x} = 5.1$ ft and the sample std deviation $s = 1.2$ ft.

Construct the 90% and 95% confidence intervals for σ^2 & σ.

(b) Suppose a sample of size $(n = 10)$ is taken from the population.

Moreover, the sample mean $\overline{x} = 5.5$ ft and the sample std deviation $s = 1.4$ ft.

Construct the 90% and 95% confidence intervals for σ^2 & σ.

(c) Which of the four confidence intervals for σ^2 from parts (a) & (b) has the most precision?