Probability: Sets, Sample Spaces, Events

Engineering Statistics
Section 2.1

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TTU

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The Need for Probability: Uncertainty in Processes

Life is full of processes whose outcome cannot be predicted ahead of time:

Definition

A **random process** is a process whose outcome cannot be predicted a priori.

Examples of random processes:

Gambling:	Flipping a Coin, Games of Chance (Blackjack, Roulette, ...)
Meteorology:	Weather Systems, Path of a Tropical Cyclone
Economics:	Stock Prices, Demand for Oil
Social Sciences:	Behaviour in People (e.g. fads)
Biology:	Behaviour of Infectious Disease
Engineering:	Instrumentation Errors, Noise in Signals
Physics:	Entropy, Heisenberg's Uncertainty Principle

If we can't predict the outcome, what's the next best thing?
Use **Probability** to determine the **likelihood** of a particular outcome!

Definition

Probability is the quantitative study of uncertainty.

Sets & Sequences (Definitions & Examples)

Sets and sequences are among the most fundamental mathematical objects:

Definition

A **set** is a possibly infinite, unique, unordered list of elements.
A **sequence** is a possibly infinite, varying, arbitrary, ordered list of elements.

Sets are written as comma-separated lists enclosed by braces.

Sequences are written as comma-separated lists enclosed by parentheses.
Finite sequences can also be written as lists w/o commas and parentheses.

Set(s):	$\{1,2,3\},\{1,2,3,4,\cdots\}$,	$\{a,b,c\},\{a,b,c,d,\cdots\}$	
Not Set(s):	$\{1,2,2\},\{1,1,1,4,\cdots\}$,	$\{a,a,c\},\{a,a,a,d,\cdots\}$	
Sequence(s):	$(1,2,3),(1,2,2)$, $(1,2,3,\cdots)$, $(1,2,2,\cdots)$	$(a,b,c),(a,b,b)$, $(a,b,c,\cdots)$, $(a,b,b,\cdots)$	abc, abb

A set can contain sets or sequences: $\{(1,2),\{a,b\},(a,c),bc,\{1,2\}\}$
Be careful with number sequences: $(1,2,3,4)=1234$ BUT $(12,34)\neq 1234$

Basic Terminology

Definition

(Experiments, Outcomes, Sample Spaces, Events)

A **random process** is a process whose outcome cannot be predicted a priori.
An **experiment** is any observation of a random process.
The **outcomes** $\omega_1, \omega_2, \ldots$ of an experiment are the different possible results.
The **sample space** Ω of an experiment is the set of all possible outcomes.
An **event** E is a subset of the sample space: $E \subseteq \Omega$

Ω is the upper-case Greek letter omega.
ω is the lower-case Greek letter omega.

REMARK: The **empty set**, $\emptyset$, is the event with no outcomes in it.
REMARK: The **empty set**, $\emptyset$, is always a subset of the sample space: $\emptyset \subseteq \Omega$

Games of Chance & Fair Objects

Some experiments involve **games of chance** such as:

- Flipping coins
- Rolling dice (of any number of sides)
- Spinning a Roulette wheel

Quite often, the coins & dice will be **fair**:

Definition

(Fair Coins & Fair Dice)

A **fair coin** has an equal chance of either side showing up upon flipping.

A **fair die** has an equal chance of any of its sides showing up upon rolling.

GOOD NEWS:
Games of chance requiring a deck of cards (blackjack, poker, ...) will not be considered in this course.

Sample Spaces, Outcomes, Events (Example)

WEX 2-1-1: Two fair coins are flipped and then their top faces are observed.

(a) Determine the sample space Ω for the experiment.

(b) Write each event as a subset of the sample space:
Event $E_1 \equiv$ Two heads occur
Event $E_2 \equiv$ A head & a tail occur
Event $E_3 \equiv$ First coin is tails

Sample Spaces, Outcomes, Events (Example)

WEX 2-1-1: Two fair coins are flipped and then their top faces are observed.

(a) Determine the sample space Ω for the experiment.
Let $H \equiv$ **H**eads AND $T \equiv$ **T**ails

(b) Write each event as a subset of the sample space:
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Let $H \equiv$ **H**eads AND $T \equiv$ **T**ails

$\boxed{\Omega = \{(H,H),\ (H,T),\ (T,H),\ (T,T)\}}$ OR $\boxed{\Omega = \{HH, HT, TH, TT\}}$

(b) Write each event as a subset of the sample space:
Event $E_1 \equiv$ Two heads occur
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$$\boxed{E_1 = \{(H,H)\}} \quad \text{OR} \quad \boxed{E_1 = \{HH\}}$$

Sample Spaces, Outcomes, Events (Example)

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$$\boxed{E_1 = \{(H,H)\}} \quad \text{OR} \quad \boxed{E_1 = \{HH\}}$$

$$\boxed{E_2 = \{(H,T),\ (T,H)\}} \quad \text{OR} \quad \boxed{E_2 = \{HT, TH\}}$$

Sample Spaces, Outcomes, Events (Example)

WEX 2-1-1: Two fair coins are flipped and then their top faces are observed.

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$$\boxed{E_1 = \{(H,H)\}} \quad \text{OR} \quad \boxed{E_1 = \{HH\}}$$

$$\boxed{E_2 = \{(H,T),\ (T,H)\}} \quad \text{OR} \quad \boxed{E_2 = \{HT, TH\}}$$

$$\boxed{E_3 = \{(T,H),\ (T,T)\}} \quad \text{OR} \quad \boxed{E_3 = \{TH, TT\}}$$

Simple & Compound Events

Definition

Event E is **simple** if it contains exactly one outcome: $E = \{\omega_1\}$

Event E is **compound** if it contains more than one outcome: $E = \{\omega_1, \ldots, \omega_k\}$

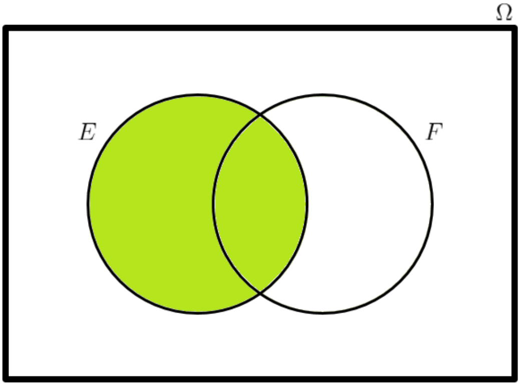
The empty set $\emptyset$ is neither a simple nor compound event.

WEX 2-1-2: Two fair coins are flipped and then their top faces are observed. Then:

Sample Space $\Omega = \{HH, HT, TH, TT\}$

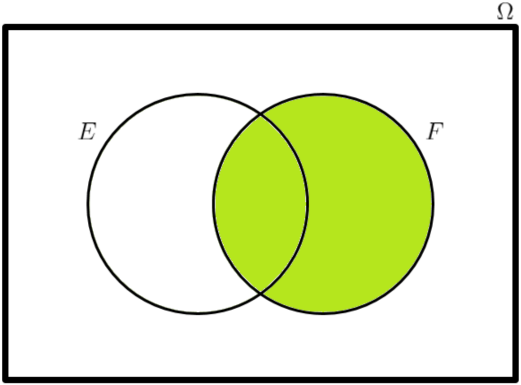
E_1	$\equiv$	Two heads occur	$=$	$\{HH\}$	$\Longrightarrow$	E_1 is simple event
E_2	$\equiv$	A head & a tail occur	$=$	$\{HT, TH\}$	$\Longrightarrow$	E_2 is compound event
E_3	$\equiv$	First coin is tails	$=$	$\{TH, TT\}$	$\Longrightarrow$	E_3 is compound event

Events (Visually via a **Venn Diagram**)



E

Events (Visually via a **Venn Diagram**)

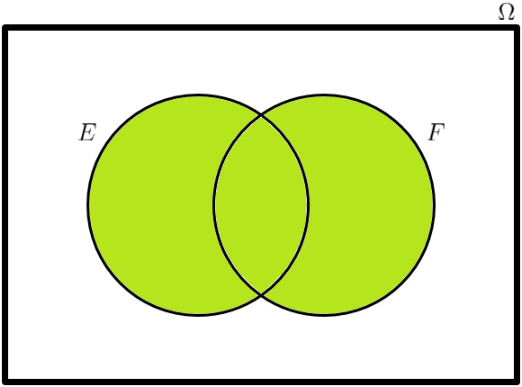


F

Union of Two Events

Definition

Let $E, F \subseteq \Omega$ be two events from sample space Ω of an experiment. Then, their **union**, $E \cup F$, is the set of all outcomes in E or F.



$E \cup F$

Inclusive OR versus Exclusive OR

For this course (and most of mathematics/statistics),
the English word "or" by itself always means **inclusive or**, never exclusive or.

Examples in English:

- Inclusive OR: "The car is compact or red" (or both compact and red)
- Exclusive OR: "I (either) drove to Austin or drove to Dallas" (but not both)

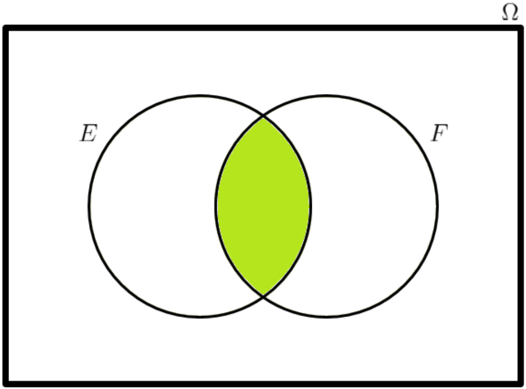
Examples in Math:

- Inclusive OR: $x > 5$ or x is divisible by 3 (or both)
- Exclusive OR: Either $x > 5$ or $x \leq 5$ (but not both)

Intersection of Two Events

Definition

Let $E, F \subseteq \Omega$ be two events from sample space Ω of an experiment. Then, their **intersection**, $E \cap F$, is the set of all outcomes in $\underline{E \text{ and } F}$.

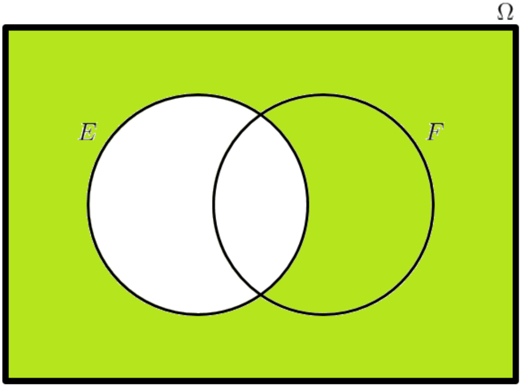


$E \cap F$

Complement of an Event

Definition

Let $E, F \subseteq \Omega$ be two events from sample space Ω of an experiment. Then, the **complement** of E, E^c, is the set of all outcomes not in E.

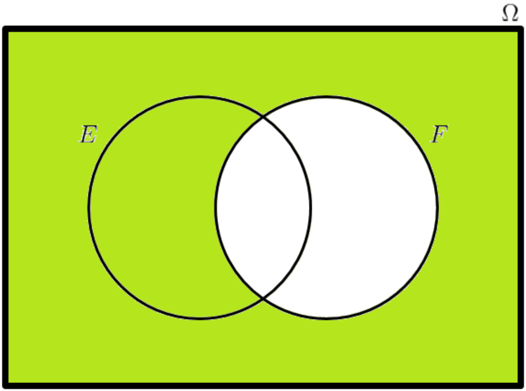


E^c

Complement of an Event

Definition

Let $E, F \subseteq \Omega$ be two events from sample space Ω of an experiment.
Then, the **complement** of F, F^c, is the set of all outcomes not in F.

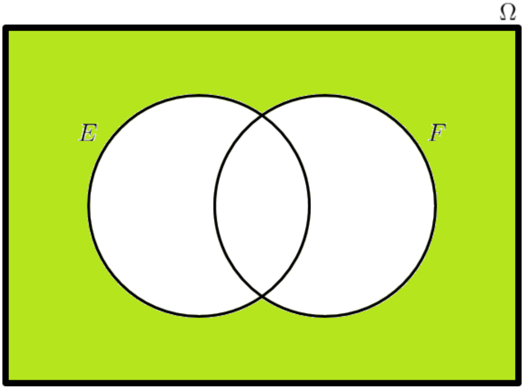


F^c

Complement of a Union of Two Events

Definition

Let $E, F \subseteq \Omega$ be two events from sample space Ω of an experiment. Then, event $(E \cup F)^c$ is the set of all outcomes neither in E nor in F.



$(E \cup F)^c$

Unions, Intersections, Complements (Example)

WEX 2-1-3: Two fair coins are flipped and then their top faces are observed. Then:

Sample Space $\Omega = \{HH, HT, TH, TT\}$

Event $E_1 \equiv$ Two heads occur $= \{HH\}$
Event $E_2 \equiv$ A head & a tail occur $= \{HT, TH\}$
Event $E_3 \equiv$ First coin is tails $= \{TH, TT\}$

$E_1 \cup E_2$	$\equiv$	(All outcomes in E_1 or E_2)	$=$	$\{HH, HT, TH\}$
$E_1 \cup E_3$	$\equiv$	(All outcomes in E_1 or E_3)	$=$	$\{HH, TH, TT\}$
$E_2 \cup E_3$	$\equiv$	(All outcomes in E_2 or E_3)	$=$	$\{HT, TH, TT\}$
$E_1 \cap E_2$	$\equiv$	(All outcomes in E_1 and E_2)	$=$	$\emptyset$
$E_1 \cap E_3$	$\equiv$	(All outcomes in E_1 and E_3)	$=$	$\emptyset$
$E_2 \cap E_3$	$\equiv$	(All outcomes in E_2 and E_3)	$=$	$\{TH\}$
E_1^c	$\equiv$	(All outcomes not in E_1)	$=$	$\{HT, TH, TT\}$
E_2^c	$\equiv$	(All outcomes not in E_2)	$=$	$\{HH, TT\}$
E_3^c	$\equiv$	(All outcomes not in E_3)	$=$	$\{HH, HT\}$
$(E_1 \cup E_2)^c$	$\equiv$	(All outcomes neither in E_1 nor in E_2)	$=$	$\{TT\}$
$(E_1 \cup E_3)^c$	$\equiv$	(All outcomes neither in E_1 nor in E_3)	$=$	$\{HT\}$
$(E_2 \cup E_3)^c$	$\equiv$	(All outcomes neither in E_2 nor in E_3)	$=$	$\{HH\}$

Unions, Intersections, Complements (Properties)

Proposition

(Properties of Unions, Intersections and Complements of Events)

Let $A, B, C \subseteq \Omega$ be events of some experiment. Then:

(*S1*)	$(A^c)^c = A$	*Complementing a Set twice is itself*
(*S2*)	$A \cup B = B \cup A$	*Commutativity of Unions*
(*S3*)	$A \cap B = B \cap A$	*Commutativity of Intersections*
(*S4*)	$(A \cup B) \cup C = A \cup (B \cup C)$	*Associativity of Unions*
(*S5*)	$(A \cap B) \cap C = A \cap (B \cap C)$	*Associativity of Intersections*
(*S6*)	$(A \cup B)^c = A^c \cap B^c$	*De Morgan's Law*
(*S7*)	$(A \cap B)^c = A^c \cup B^c$	*De Morgan's Law*
(*S8*)	$A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset$	*Unions/Intersections with Empty Set*
(*S9*)	$\Omega^c = \emptyset, \quad \emptyset^c = \Omega$	*Sample Space & Empty Set Relation*
(*S10*)	$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$	*Intersection Distributes over Union*
(*S11*)	$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$	*Union Distributes over Intersection*

PROOF: Use Venn Diagrams or take **Intro to Proof**. (MATH 3310)

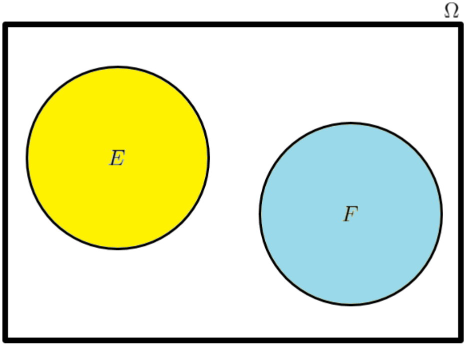
Mutual Exclusivity of Two Events (Definition)

Definition

Events E, F are **mutually exclusive** if they have no outcomes in common:

$$E \cap F = \emptyset$$

Events E & F in this case are sometimes called **disjoint** events.



Mutually Exclusive Events

Mutual Exclusivity of Two Events (Definition)

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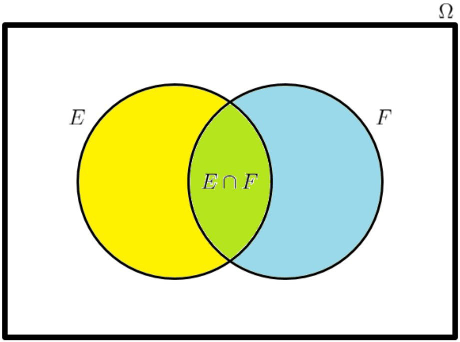

Not Mutually Exclusive

Mutual Exclusivity of Two Events (Example)

WEX 2-1-4: Two fair coins are flipped and then their top faces are observed. Then:

Sample Space $\Omega = \{HH, HT, TH, TT\}$

Event $E_1 \equiv$ Two heads occur $= \{HH\}$
Event $E_2 \equiv$ A head & a tail occur $= \{HT, TH\}$
Event $E_3 \equiv$ First coin is tails $= \{TH, TT\}$

$$E_1 \cap E_2 = \emptyset \implies E_1 \text{ and } E_2 \text{ are mutually exclusive}$$
$$E_1 \cap E_3 = \emptyset \implies E_1 \text{ and } E_3 \text{ are mutually exclusive}$$
$$E_2 \cap E_3 = \{TH\} \implies E_2 \text{ and } E_3 \text{ are } \underline{\text{not}} \text{ mutually exclusive}$$

Mutual Exclusivity of Three Events (Definition)

Mutual Exclusivity can be extended to three events:

Definition

Events E, F, G are **mutually exclusive** if they have no outcomes in common:

$$E \cap F = \emptyset \text{ and } E \cap G = \emptyset \text{ and } F \cap G = \emptyset$$

Events E, F, G in this case are also called **pairwise disjoint** events.

QUESTION TO PONDER:

Why are mutually exclusive events E, F, G not defined to be $E \cap F \cap G = \emptyset$???

Mutual Exclusivity of Many Events (Definition)

Mutual Exclusivity can be extended to many events:

Definition

Events $E_1, E_2, \ldots, E_n$ are **mutually exclusive** if they have no outcomes in common:

$$E_i \cap E_j = \emptyset \text{ for } i \neq j$$

Events $E_1, E_2, \ldots, E_n$ in this case are also called **pairwise disjoint** events.

QUESTION TO PONDER:

Why is the definition for mutual exclusivity not $E_1 \cap E_2 \cap \cdots \cap E_n = \emptyset$???

Textbook Logistics for Section 2.1

- Difference(s) in Terminology:

TEXTBOOK TERMINOLOGY	SLIDES/OUTLINE TERMINOLOGY
Null Event $\emptyset$	Empty Set $\emptyset$

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Sample Space	$\mathcal{S}$	Ω
Complement of Event	A'	A^c

Fin

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