Probability: Tuples, Permutations, Combinations

Engineering Statistics
Section 2.3

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Probability: Equally Likely Outcomes

Very often, all the outcomes of an experiment are **equally likely to occur**:

Definition

(Probability of an Event)

Let Ω be the sample space of an experiment with **equally likely outcomes**.
Let E be an event of the experiment.

Then the **probability** of event E occurring is defined as: $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$

i.e. The probability is the proportion of outcomes that comprise the event.

REMARKS:
Often it's impractical to list every outcome of a sample space Ω or event E.
When computing probability, only the **measures** of E & Ω are needed.

QUESTION: How to easily count all outcomes in large events???
ANSWER: Combinatorics!

Definition

Combinatorics is the study of sophisticated counting.

PART I: MULTI-STAGE EXPERIMENTS, TUPLES

Multi-Stage Experiments

Some experiments are formed in **stages**:

Definition

A **multi-stage experiment** is an experiment that can be rephrased as one with several stages separated by the phrase ”and then” or ”followed by”.

Examples of Multi-Stage Experiments:

- Flipping two fair coins:

  ”Flip a fair coin, and then flip another fair coin”

- Forming a 3-person committee out of 50 people:

  ”Randomly select a person, and then randomly select a remaining person, followed by randomly selecting a remaining person”

Sets, Sequences, Tuples

Sets, sequences & tuples are among the most fundamental math objects:

Definition

A **set** is a possibly infinite, unique, unordered list of elements.
A **sequence** is a possibly infinite, varying, arbitrary, ordered list of elements.
A k-**tuple** is a finite sequence with k elements.

Ordered pairs are 2-tuples. **Ordered triples** are 3-tuples.

Sets:	$\{1,2,3\}, \{1,2,3,4,\cdots\}$,	$\{a,b,c\}, \{a,b,c,d,\cdots\}$	
Not Sets:	$\{1,2,2\}, \{1,1,1,4,\cdots\}$,	$\{a,a,c\}, \{a,a,a,d,\cdots\}$	
Sequences:	$(1,2,3), (1,2,2)$, $(1,2,3,\cdots)$, $(1,2,2,\cdots)$	$(a,b,c), (a,b,b)$, $(a,b,c,\cdots)$, $(a,b,b,\cdots)$	abc, abb
Tuples:	$(1,2), (1,2,3), (1,2,3,4)$,	$(a,b), (a,b,c), (a,b,c,d)$,	ab, abc
Not Tuples:	$(1,2,3,\cdots)$,	$(a,b,c,\cdots)$	

A set can contain sets or sequences: $\{(1,2), \{a,b\}, (a,c), bc, \{1,2\}\}$
Be careful with number sequences: $(1,2,3,4) = 1234$ BUT $(12,34) \neq 1234$

Counting Tuples

Proposition

(Counting Tuples)

Given a set of k-tuples such that:

$$\begin{array}{lcccc}
\textit{there are} & n_1 & \textit{possible choices for the} & 1^{st} & \textit{element,} \\
\textit{there are} & n_2 & \textit{possible choices for the} & 2^{nd} & \textit{element,} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\textit{there are} & n_{k-1} & \textit{possible choices for the} & (k-1)^{st} & \textit{element,} \\
\textit{there are} & n_k & \textit{possible choices for the} & k^{th} & \textit{element.}
\end{array}$$

*Then the # of possible k**-tuples** is:* $\displaystyle\prod_{j=1}^{k} n_j = n_1 n_2 \cdots n_{k-1} n_k$

Counting Tuples (from Multi-Stage Experiments)

- Flipping two fair coins:

  "Flip a fair coin, and then flip another fair coin"

  The first coin flip has two possible results (Heads or Tails), and the second coin flip has two possible results (Heads or Tails).
  Therefore, there are $(2)(2) = 4$ 2-tuples (ordered pairs):

  $(H,H),\ (H,T),\ (T,H),\ (T,T)$ OR $HH,\ HT,\ TH,\ TT$

- Forming a 3-person committee out of 50 people:

  "Randomly select a person, and then randomly select a remaining person, followed by randomly selecting a remaining person"

  The 1^{st} selection is from 50 people, the 2^{nd} selection is from 49 people, and the 3^{rd} selection is from 48 people.
  Therefore, there are $(50)(48)(49) = 117,600$ 3-tuples (ordered triples)...
  ...which are too tedious to list out in their entirety, but here's a few:

  (Bob,Joe,Sally), (Jane,Joe,Bob), (Sally,Mike,Jane), ...

PART II: PERMUTATIONS, COMBINATIONS

Factorials

Recall the definition of **factorial** of a non-negative integer:

Definition

(Factorial)

Let k be a positive integer. Then $k! := k(k-1)(k-2)\cdots(4)(3)(2)(1)$

To avoid corner cases, $0! := 1$

$$
\begin{array}{rcccc}
0! & = & 1 & = & 1 \\
1! & = & (1) & = & 1 \\
2! & = & (2)(1) & = & 2 \\
3! & = & (3)(2)(1) & = & 6 \\
4! & = & (4)(3)(2)(1) & = & 24 \\
5! & = & (5)(4)(3)(2)(1) & = & 120 \\
\vdots & & \vdots & & \vdots
\end{array}
$$

Factorials are used in developing permutations & combinations.

Definition

A k-**permutation** of an n-element set is a finite sequence (i.e. tuple) with k distinct elements such that **order matters**.

For instance, given the following 4-element set $\{a, b, c, d\}$, then:

1-permutations:	a, b, c, d
2-permutations:	$ab, ac, ad,$ $ba, bc, bd,$ $ca, cb, cd,$ da, db, dc
3-permutations:	$abc, abd, acb, acd, adb, adc,$ $bac, bad, bca, bcd, bda, bdc,$ $cab, cad, cba, cbd, cda, cdb,$ $dab, dac, dba, dbc, dca, dcb$
4-permutations:	$abcd, abdc, acbd, acdb, adbc, adcb,$ $bacd, badc, bcad, bcda, bdac, bdca,$ $cabd, cadb, cbad, cbda, cdab, cdba,$ $dabc, dacb, dbac, dbca, dcab, dcba$

For instance, given the following 4-element set $\{a, b, c, d\}$, then:

2-permutations:	$ab, ac, ad,$ $ba, bc, bd,$ $ca, cb, cd,$ da, db, dc

Q: How to count the # of 2-permutations without explicitly writing all them out?

A: Construct a fitting multi-stage experiment for this problem!!

Experiment: Choose an element from set for 1^{st} position, and then choose a remaining element for 2^{nd} position.

There are 4 elements from set available for 1^{st} position and then there are 3 elements available for 2^{nd} position.

Therefore, there are $(4)(3) = 12$ 2-permutations of the 4-element set.

Counting Permutations (Motivation)

For instance, given the following 4-element set $\{a, b, c, d\}$, then:

3-permutations:	$abc, abd, acb, acd, adb, adc,$ $bac, bad, bca, bcd, bda, bdc,$ $cab, cad, cba, cbd, cda, cdb,$ $dab, dac, dba, dbc, dca, dcb$

Q: How to count the # of 3-permutations <u>without explicitly writing all them out</u>?

A: Construct a fitting multi-stage experiment for this problem!!

Experiment: Choose an element from set for 1^{st} position, and then choose a remaining element for 2^{nd} position, and then choose a remaining element for 3^{rd} position.

There are 4 elements from set available for 1^{st} position and then there are 3 elements available for 2^{nd} position and then there are 2 elements available for 3^{rd} position.

Therefore, there are $(4)(3)(2) = 24$ 3-permutations of the 4-element set.

Counting Permutations (Motivation)

For instance, given the following 4-element set $\{a, b, c, d\}$, then:

4-permutations:	$abcd, abdc, acbd, acdb, adbc, adcb,$ $bacd, badc, bcad, bcda, bdac, bdca,$ $cabd, cadb, cbad, cbda, cdab, cdba,$ $dabc, dacb, dbac, dbca, dcab, dcba$

Q: How to count the # of 4-permutations without explicitly writing all them out?

A: Construct a fitting multi-stage experiment for this problem!!

Experiment: Choose an element from set for 1^{st} position, and then choose a remaining element for 2^{nd} position, and then choose a remaining element for 3^{rd} position, and then choose a remaining element for 4^{th} position.

There are 4 elements from set available for 1^{st} position and then there are 3 elements available for 2^{nd} position and then there are 2 elements available for 3^{rd} position and then there is 1 element available for 4^{th} position.

Therefore, there are $(4)(3)(2)(1) = 24$ 4-permutations of the 4-element set.

Counting Permutations

So, the # of k-permutations of an n-element set is:

$$n(n-1)(n-2)\cdots[n-(k-1)]$$

$$= \frac{n(n-1)(n-2)\cdots[n-(k-1)](n-k)[n-(k+1)]\cdots(3)(2)(1)}{(n-k)[n-(k+1)]\cdots(3)(2)(1)}$$

$$:= \frac{n!}{(n-k)!}$$

Proposition

(Counting Permutations)

Let k, n be integers such that $0 \leq k \leq n$.

*Then the # of k-**permutations of an** n-**element set** is:* $P_k^n := \dfrac{n!}{(n-k)!}$

Combinations (Definition)

Definition

A k-**combination** of an n-element set is a finite sequence (i.e. tuple) with k distinct elements such that **order does not matter**.

For instance, given the following 4-element set $\{a, b, c, d\}$, then:

1-combinations:	a, b, c, d
2-combinations:	$ab, ac, ad,$ $bc, bd,$ cd
3-combinations:	$abc, abd, acd,$ bcd
4-combinations:	$abcd$

Comparison of Permutations & Combinations

For instance, given the following 4-element set $\{a, b, c, d\}$, then:

2-permutations:	$ab, ac, ad,$ $ba, bc, bd,$ $ca, cb, cd,$ da, db, dc
3-permutations:	$abc, abd, acb, acd, adb, adc,$ $bac, bad, bca, bcd, bda, bdc,$ $cab, cad, cba, cbd, cda, cdb,$ $dab, dac, dba, dbc, dca, dcb$
4-permutations:	$abcd, abdc, acbd, acdb, adbc, adcb,$ $bacd, badc, bcad, bcda, bdac, bdca,$ $cabd, cadb, cbad, cbda, cdab, cdba,$ $dabc, dacb, dbac, dbca, dcab, dcba$

2-combinations:	$ab, ac, ad,$ $bc, bd,$ cd
3-combinations:	$abc, abd, acd,$ bcd
4-combinations:	$abcd$

For instance, given the following 4-element set $\{a, b, c, d\}$, then:

1-combinations:	a, b, c, d
2-combinations:	$ab, ac, ad,$ $bc, bd,$ cd
3-combinations:	$abc, abd, acd,$ bcd
4-combinations:	$abcd$

For each k-combination, there are $k!$ k-permutations of that k-combination. But with combinations, order does not matter, so only one of the k-permutations count as a k-combination.

Hence, k-permutations & k-combinations are related as follows:

$$(\text{\# of } k\text{-permutations}) = k! \times (\text{\# of } k\text{-combinations})$$

so solving for the # of k-combinations yields

$$(\text{\# of } k\text{-combinations}) = \frac{1}{k!} \times (\text{\# of } k\text{-permutations})$$

Counting Combinations

For each k-combination, there are $k!$ k-permutations of that k-combination. But with combinations, order does not matter, so only one of the k-permutations count as a k-combination.

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so solving for the # of k-combinations yields

$$(\text{\# of } k\text{-combinations}) = \frac{1}{k!} \times (\text{\# of } k\text{-permutations})$$

Proposition

(Counting Combinations)

Let k, n be integers such that $0 \leq k \leq n$.

Then the # of k ***-combinations of an*** *n* ***-element set*** *is:* $\binom{n}{k} := \dfrac{n!}{k!(n-k)!}$

Combinations & Pascal's Triangle

Recall from Algebra that **Pascal's Triangle** is useful in expanding $(x+y)^n$:

$$
\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & & \leftarrow n=0 \\
 & & & & 1 & & 1 & & & & & & \leftarrow n=1 \\
 & & & 1 & & 2 & & 1 & & & & & \leftarrow n=2 \\
 & & 1 & & 3 & & 3 & & 1 & & & & \leftarrow n=3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & & \leftarrow n=4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \leftarrow n=5 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n$

Notice that the coefficient of each term is a combination.

Examples: $\binom{2}{0} = 1, \quad \binom{3}{3} = 1, \quad \binom{5}{2} = 10$

Advanced Counting

Permutations & combinations just scratch the surface of advanced counting:

- Stirling Numbers
- Catalan Numbers
- Recurrence Relations
- Generating Functions
- Counting Problems involving Cycles
- Counting Problems involving Geometric Symmetry

To learn more, take:

- **Discrete Math** (CS 1382)
- **Combinatorics** (MATH 4363)

Textbook Logistics for Section 2.3

- Difference(s) in Terminology:

TEXTBOOK TERMINOLOGY	SLIDES/OUTLINE TERMINOLOGY
Null Event $\emptyset$	Empty Set $\emptyset$
Number of Outcomes in E	Measure of E

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Sample Space	$\mathcal{S}$	Ω
Complement of Event	A'	A^c
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Measure of Event	$N(A)$	$\lvert A \rvert$
k-Permutations of n-element Set	$P_{k,n}$	P_k^n

The Good News!!

As one can tell, counting is surprising tricky and difficult!!

Here's the good news:

1. THE HOMEWORK FOR THIS SECTION IS PURELY BONUS (AND, THEREFORE, IS OPTIONAL)
2. ANY EXAM PROBLEMS INVOLVING PERMUTATIONS & COMBINATIONS WILL BE BONUS!!

Fin.