Probability: Conditioning, Bayes' Theorem

Engineering Statistics
Section 2.4

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TTU

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PART I: CONDITIONAL PROBABILITY

Conditional Probability (Definition)

The occurrence of one event can affect the probability of another event:

Definition

(Conditional Probability)

Let events E, F be events in the sample space Ω of an experiment.

Then:

- The **conditional probability** of F given E, denoted $\mathbb{P}(F|E)$, is the probability of event F assuming that event E has already occurred.
- The **conditional probability** of E given F, denoted $\mathbb{P}(E|F)$, is the probability of event E assuming that event F has already occurred.

WARNING: **Order matters**: in general, $\mathbb{P}(F|E) \neq \mathbb{P}(E|F)$

Conditional Probability

But the previous definition is too crude to use.
How does conditional probability relate to ordinary probability?

Proposition

(Conditional Probability)

Let events E, F be events in the sample space Ω such that $|E| > 0$.

Then:

$$\mathbb{P}(\textit{If } E \textit{ then } F) = \mathbb{P}(F \textit{ given } E) = \mathbb{P}(F|E) := \frac{|E \cap F|}{|E|}$$

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Let events E, F be events in the sample space Ω such that $\mathbb{P}(E) > 0$.

Then:

$$\mathbb{P}(\textit{If } E \textit{ then } F) = \mathbb{P}(F \textit{ given } E) = \mathbb{P}(F|E) := \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)}$$

WARNING: **Order matters**: in general, $\mathbb{P}(F|E) \neq \mathbb{P}(E|F)$

PROOF:

$$\mathbb{P}(F|E) = \frac{|E \cap F|}{|E|} = \frac{|E \cap F|/|\Omega|}{|E|/|\Omega|} = \frac{|E \cap F|}{|\Omega|} \div \frac{|E|}{|\Omega|} = \mathbb{P}(E \cap F) \div \mathbb{P}(E) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)}$$

Conditional Probability (Example)

<u>WEX 2-4-1:</u> A fair coin is flipped and a fair six-sided die is rolled.
(a) Find the probability that the coin shows tails given that the die shows 5.
(b) Find the probability that if the coin shows tails then the die shows 5.

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Sample space $\Omega = \left\{ \begin{array}{c} (H,1),(H,2),(H,3),(H,4),(H,5),(H,6), \\ (T,1),(T,2),(T,3),(T,4),(T,5),(T,6) \end{array} \right\}$

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Let event $E \equiv$ (Coin shows tails) $= \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$
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Then $E \cap F = \{(T,5)\}$

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$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{6}{12} = \frac{1}{2}, \quad \mathbb{P}(F) = \frac{|F|}{|\Omega|} = \frac{2}{12} = \frac{1}{6}, \quad \mathbb{P}(E \cap F) = \frac{|E \cap F|}{|\Omega|} = \frac{1}{12}$$

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Intersection of Events (Alternative Formula)

The intersection of two events can found using conditional probability:

Proposition

(Intersection of Two Events)

Let events E, F be events in the sample space Ω of an experiment.

Then:

$$\mathbb{P}(E \cap F) = \mathbb{P}(E) \cdot \mathbb{P}(F|E)$$

or equivalently

$$\mathbb{P}(E \cap F) = \mathbb{P}(F) \cdot \mathbb{P}(E|F)$$

PROOF: Solve $\mathbb{P}(F|E) = \dfrac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)}$ or $\mathbb{P}(E|F) = \dfrac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$ for $\mathbb{P}(E \cap F)$. $\square$

PART II: PARTITIONS OF SAMPLE SPACE

LAW OF TOTAL PROBABILITY, BAYES' THEOREM

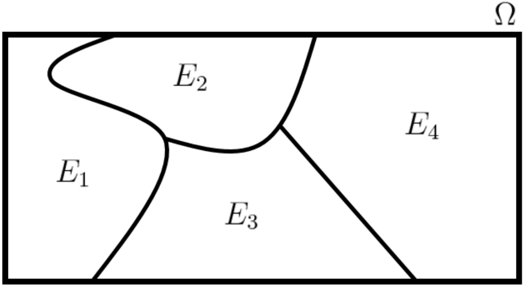
Partition of the Sample Space

Definition

Events $E_1, E_2, \ldots, E_k \subseteq \Omega$ **partition** sample space Ω if:

$$E_1, E_2, \ldots, E_k \text{ are } \underline{\text{pairwise disjoint}} \quad \text{AND} \quad \bigcup_{i=1}^{k} E_i = \Omega$$

Think of sample space as a puzzle & the partitioning events as puzzle pieces.



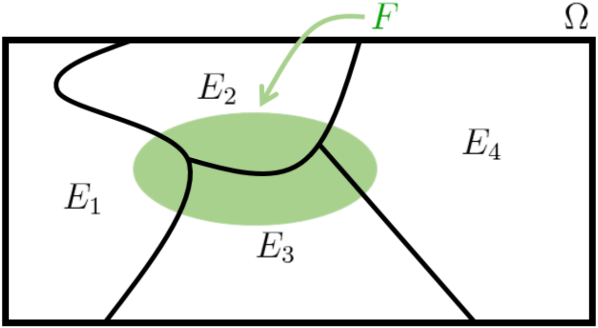
Events E_1, E_2, E_3, E_4 partition sample space Ω.

Law of Total Probability (LTP)

Theorem

(Law of Total Probability)

Let $E_1, \ldots, E_k$ *partition sample space* Ω. *Then* $\mathbb{P}(F) = \sum_{i=1}^{k} \mathbb{P}(F|E_i) \cdot \mathbb{P}(E_i)$



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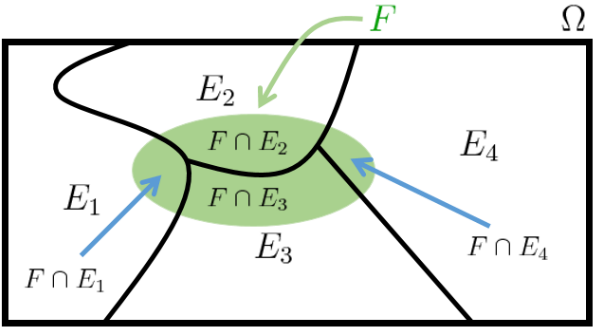

PROOF: Observe that events $F \cap E_1, F \cap E_2, \ldots, F \cap E_k$ are pairwise disjoint since events $E_1, E_2, \ldots, E_k$ are pairwise disjoint.

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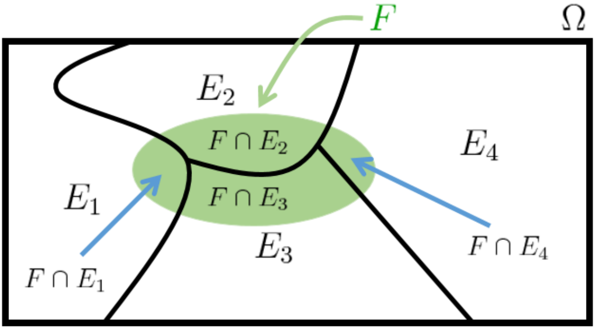


PROOF: Then, $\mathbb{P}(F) = \sum_{i=1}^{k} \mathbb{P}(F \cap E_i) = \sum_{i=1}^{k} \mathbb{P}(F|E_i) \cdot \mathbb{P}(E_i)$ □

Bayes' Theorem

Theorem

(Bayes' Theorem)

Let events $E_1, \ldots, E_k \subseteq \Omega$ *partition sample space* Ω. *Then*

$$\mathbb{P}(E_j|F) = \frac{\mathbb{P}(F|E_j)\cdot\mathbb{P}(E_j)}{\displaystyle\sum_{i=1}^{k}\mathbb{P}(F|E_i)\cdot\mathbb{P}(E_i)} \quad \textit{for} \quad j = 1, 2, \ldots, k$$

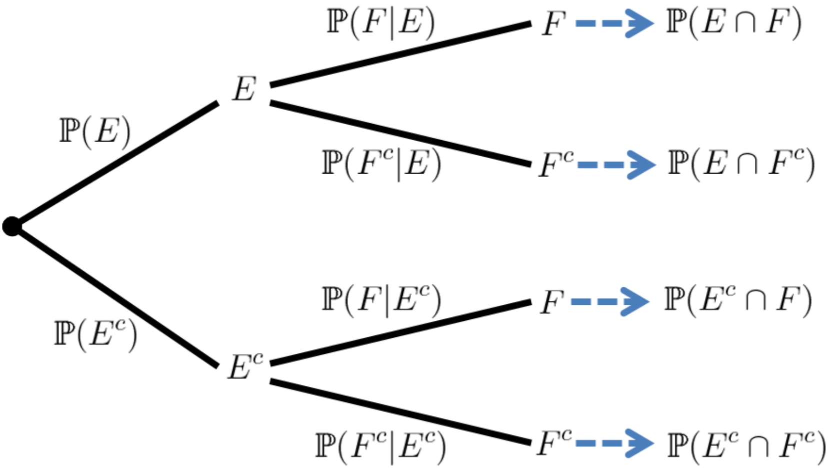
PROOF: $\mathbb{P}(E_j|F) = \dfrac{\mathbb{P}(F\cap E_j)}{\mathbb{P}(F)} = \dfrac{\mathbb{P}(F|E_j)\cdot\mathbb{P}(E_j)}{\mathbb{P}(F)} \overset{LTP}{=} \dfrac{\mathbb{P}(F|E_j)\cdot\mathbb{P}(E_j)}{\sum_{i=1}^{k}\mathbb{P}(F|E_i)\cdot\mathbb{P}(E_i)}$ □

PART III

PART III: PROBABILITIY TREES & JOINT PROBABILITY TABLES

Two-Stage Experiments & Probability Trees

Two-stage experiments can be visualized using a **probability tree**:



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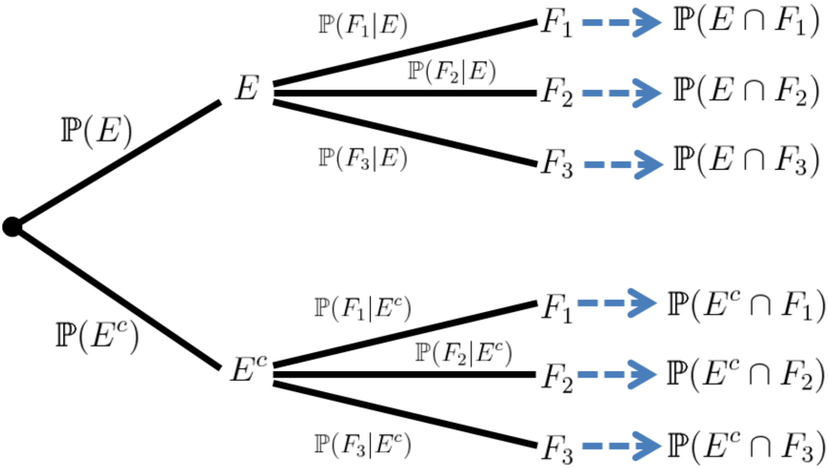

Events F_1, F_2, F_3 must partition the sample space Ω.

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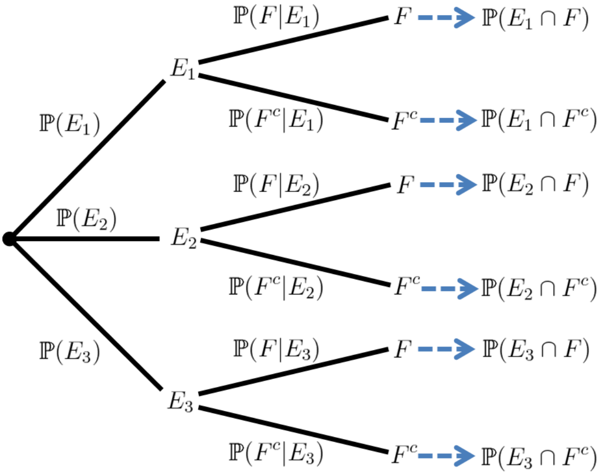

Events E_1, E_2, E_3 must partition the sample space Ω.

2-stage experiments can be summarized using 2-way **joint probability table**:

	F	F^c	**TOTAL**
E	$\mathbb{P}(E \cap F)$	$\mathbb{P}(E \cap F^c)$	$\mathbb{P}(E)$
E^c	$\mathbb{P}(E^c \cap F)$	$\mathbb{P}(E^c \cap F^c)$	$\mathbb{P}(E^c)$
TOTAL	$\mathbb{P}(F)$	$\mathbb{P}(F^c)$	(**DON'T CARE**)

	F_1	F_2	F_2	**TOTAL**
E_1	$\mathbb{P}(E_1 \cap F_1)$	$\mathbb{P}(E_1 \cap F_2)$	$\mathbb{P}(E_1 \cap F_3)$	$\mathbb{P}(E_1)$
E_2	$\mathbb{P}(E_2 \cap F_1)$	$\mathbb{P}(E_2 \cap F_2)$	$\mathbb{P}(E_2 \cap F_3)$	$\mathbb{P}(E_2)$
E_3	$\mathbb{P}(E_3 \cap F_1)$	$\mathbb{P}(E_3 \cap F_2)$	$\mathbb{P}(E_3 \cap F_3)$	$\mathbb{P}(E_3)$
TOTAL	$\mathbb{P}(F_1)$	$\mathbb{P}(F_2)$	$\mathbb{P}(F_3)$	(**DON'T CARE**)

Here, events E_1, E_2, E_3 must partition sample space Ω.
Here, events F_1, F_2, F_3 must partition sample space Ω.

3-stage experiments can be summarized via 3-way **joint probability tables**:

E	G	G^c	**TOTAL**
F	$\mathbb{P}(E \cap F \cap G)$	$\mathbb{P}(E \cap F \cap G^c)$	$\mathbb{P}(E \cap F)$
F^c	$\mathbb{P}(E \cap F^c \cap G)$	$\mathbb{P}(E \cap F^c \cap G^c)$	$\mathbb{P}(E \cap F^c)$
TOTAL	$\mathbb{P}(E \cap G)$	$\mathbb{P}(E \cap G^c)$	(**DON'T CARE**)

E^c	G	G^c	**TOTAL**
F	$\mathbb{P}(E^c \cap F \cap G)$	$\mathbb{P}(E^c \cap F \cap G^c)$	$\mathbb{P}(E^c \cap F)$
F^c	$\mathbb{P}(E^c \cap F^c \cap G)$	$\mathbb{P}(E^c \cap F^c \cap G^c)$	$\mathbb{P}(E^c \cap F^c)$
TOTAL	$\mathbb{P}(E^c \cap G)$	$\mathbb{P}(E^c \cap G^c)$	(**DON'T CARE**)

3-Stage Experiments & 3-Way Joint Probability Tables

E_1	G_1	G_2	G_3	**TOTAL**
F_1	$\mathbb{P}(E_1 \cap F_1 \cap G_1)$	$\mathbb{P}(E_1 \cap F_1 \cap G_2)$	$\mathbb{P}(E_1 \cap F_1 \cap G_3)$	$\mathbb{P}(E_1 \cap F_1)$
F_2	$\mathbb{P}(E_1 \cap F_2 \cap G_1)$	$\mathbb{P}(E_1 \cap F_2 \cap G_2)$	$\mathbb{P}(E_1 \cap F_2 \cap G_3)$	$\mathbb{P}(E_1 \cap F_2)$
F_3	$\mathbb{P}(E_1 \cap F_3 \cap G_1)$	$\mathbb{P}(E_1 \cap F_3 \cap G_2)$	$\mathbb{P}(E_1 \cap F_3 \cap G_3)$	$\mathbb{P}(E_1 \cap F_3)$
TOTAL	$\mathbb{P}(E_1 \cap G_1)$	$\mathbb{P}(E_1 \cap G_2)$	$\mathbb{P}(E_1 \cap G_3)$	(**DON'T CARE**)

E_2	G_1	G_2	G_3	**TOTAL**
F_1	$\mathbb{P}(E_2 \cap F_1 \cap G_1)$	$\mathbb{P}(E_2 \cap F_1 \cap G_2)$	$\mathbb{P}(E_2 \cap F_1 \cap G_3)$	$\mathbb{P}(E_2 \cap F_1)$
F_2	$\mathbb{P}(E_2 \cap F_2 \cap G_1)$	$\mathbb{P}(E_2 \cap F_2 \cap G_2)$	$\mathbb{P}(E_2 \cap F_2 \cap G_3)$	$\mathbb{P}(E_2 \cap F_2)$
F_3	$\mathbb{P}(E_2 \cap F_3 \cap G_1)$	$\mathbb{P}(E_2 \cap F_3 \cap G_2)$	$\mathbb{P}(E_2 \cap F_3 \cap G_3)$	$\mathbb{P}(E_2 \cap F_3)$
TOTAL	$\mathbb{P}(E_2 \cap G_1)$	$\mathbb{P}(E_2 \cap G_2)$	$\mathbb{P}(E_2 \cap G_3)$	(**DON'T CARE**)

E_3	G_1	G_2	G_3	**TOTAL**
F_1	$\mathbb{P}(E_3 \cap F_1 \cap G_1)$	$\mathbb{P}(E_3 \cap F_1 \cap G_2)$	$\mathbb{P}(E_3 \cap F_1 \cap G_3)$	$\mathbb{P}(E_3 \cap F_1)$
F_2	$\mathbb{P}(E_3 \cap F_2 \cap G_1)$	$\mathbb{P}(E_3 \cap F_2 \cap G_2)$	$\mathbb{P}(E_3 \cap F_2 \cap G_3)$	$\mathbb{P}(E_3 \cap F_2)$
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TOTAL	$\mathbb{P}(E_3 \cap G_1)$	$\mathbb{P}(E_3 \cap G_2)$	$\mathbb{P}(E_3 \cap G_3)$	(**DON'T CARE**)

Events E_1, E_2, E_3 must partition Ω. Ditto for F_1, F_2, F_3. Ditto for G_1, G_2, G_3.

Some Loose Guidelines to avoid Dead-Ends

How does one choose among using measures, probability trees, and joint probability tables???

GIVEN SITUATION	METHOD TO USE
Small Sample Space and its Outcomes	$\mathbb{P}(E\|F) = \frac{\|E \cap F\|}{\|F\|}$
Many Ordinary Probabilities	$\mathbb{P}(E\|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$
Many Conditional Probabilities	Probability Tree
Many Intersection Probabilities	Joint Probability Table

NOTE: Using set operation properties may be necessary.

NOTE: Constructing a Venn Diagram may be helpful.

NOTE: Using the Principle of Inclusion-Exclusion may be helpful/necessary.

Textbook Logistics for Section 2.4

- Difference(s) in Terminology:

TEXTBOOK TERMINOLOGY	SLIDES/OUTLINE TERMINOLOGY
Null Event $\emptyset$	Empty Set $\emptyset$
Number of Outcomes in E	Measure of E

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Sample Space	$\mathcal{S}$	Ω
Complement of Event	A'	A^c
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Measure of Event	$N(A)$	$\lvert A\rvert$
k-Permutations of n-element Set	$P_{k,n}$	P_k^n
Conditional Probability	$P(A\lvert B)$	$\mathbb{P}(A\lvert B)$

Fin.