Probability: Independence of Events

Engineering Statistics
Section 2.5

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10 February 2016

Independence of Two Events

Sometimes one would like to know if the occurrence of one event affects the chances of another event occurring.

Definition

Let $E, F \subseteq \Omega$ be two events from sample space Ω of an experiment.

Then E & F are **independent** if $\mathbb{P}(E|F) = \mathbb{P}(E)$.

Otherwise, E & F are **dependent** if $\mathbb{P}(E|F) \neq \mathbb{P}(E)$.

WARNING: Two pairwise disjoint events are never independent!!!

Proposition

Events E & F are independent if and only if $\mathbb{P}(E \cap F) = \mathbb{P}(E) \cdot \mathbb{P}(F)$

PROOF: Let E & F be independent. Then, by definition, $\mathbb{P}(E|F) = \mathbb{P}(E)$.
Hence, $\mathbb{P}(E \cap F) = \mathbb{P}(F) \cdot \mathbb{P}(E|F) \stackrel{IND}{=} \mathbb{P}(F) \cdot \mathbb{P}(E)$ $\square$

Independence of Two Events (Example)

WEX 2-5-1: Two fair coins are flipped and then their top faces are observed.

Sample Space $\Omega = \{HH, HT, TH, TT\}$

Let Event E	$\equiv$	1^{st} coin is heads	$=$	$\{HH,\ HT\}$	$\Longrightarrow$	$\mathbb{P}(E) = 0.5$
Let Event F	$\equiv$	2^{nd} coin is tails	$=$	$\{HT,\ TT\}$	$\Longrightarrow$	$\mathbb{P}(F) = 0.5$
Let Event G	$\equiv$	Both coins are heads	$=$	$\{HH\}$	$\Longrightarrow$	$\mathbb{P}(G) = 0.25$

Then, $E \cap F = \{HT\} \Longrightarrow \mathbb{P}(E \cap F) = 0.25$ and $\mathbb{P}(E) \cdot \mathbb{P}(F) = (0.5)(0.5) = 0.25$

$\therefore\ \mathbb{P}(E \cap F) = \mathbb{P}(E) \cdot \mathbb{P}(F) \Longrightarrow$ Events E & F are independent

But, $E \cap G = \{HH\} \Longrightarrow \mathbb{P}(E \cap G) = 0.25$ and $\mathbb{P}(E) \cdot \mathbb{P}(G) = (0.5)(0.25) = 0.125$

$\therefore\ \mathbb{P}(E \cap G) \neq \mathbb{P}(E) \cdot \mathbb{P}(G) \Longrightarrow$ Events E & G are not independent

$F \cap G = \emptyset \Longrightarrow \mathbb{P}(F \cap G) = 0$ and $\mathbb{P}(F) \cdot \mathbb{P}(G) = (0.5)(0.25) = 0.125$

$\therefore\ \mathbb{P}(F \cap G) \neq \mathbb{P}(F) \cdot \mathbb{P}(G) \Longrightarrow$ Events F & G are not independent

Notice that F & G are pairwise disjoint and, yet, are not independent!!

Independence of Complement(s) of Two Events

Often, it's crucial to know if complement(s) of event(s) are independent or not:

Proposition

Events E & F are independent if and only if E^c & F^c are independent.

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PROOF: Let E & F be independent. Then $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$.

$$\begin{aligned} \mathbb{P}(E^c \cap F^c) &= \mathbb{P}[(E \cup F)^c] \\ &= 1 - \mathbb{P}(E \cup F) \\ &= 1 - \mathbb{P}(E) - \mathbb{P}(F) + \mathbb{P}(E \cap F) \\ &\overset{IND}{=} 1 - \mathbb{P}(E) - \mathbb{P}(F) + \mathbb{P}(E)\mathbb{P}(F) \\ &= [1 - \mathbb{P}(E)][1 - \mathbb{P}(F)] \\ &= \mathbb{P}(E^c)\mathbb{P}(F^c) \end{aligned}$$

$\therefore$ E^c & F^c are independent □

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PROOF: Let E & F be independent. Then $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$.

$$\begin{aligned} \mathbb{P}(E^c \cap F) &= \mathbb{P}(F) - \mathbb{P}(E \cap F) \\ &\overset{IND}{=} \mathbb{P}(F) - \mathbb{P}(E)\mathbb{P}(F) \\ &= [1 - \mathbb{P}(E)]\mathbb{P}(F) \\ &= \mathbb{P}(E^c)\mathbb{P}(F) \end{aligned}$$

$\therefore$ E^c & F are independent □

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PROOF: Let E & F be independent. Then $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$.

$$\begin{aligned} \mathbb{P}(E \cap F^c) &= \mathbb{P}(E) - \mathbb{P}(E \cap F) \\ &\overset{IND}{=} \mathbb{P}(E) - \mathbb{P}(E)\mathbb{P}(F) \\ &= \mathbb{P}(E)[1 - \mathbb{P}(F)] \\ &= \mathbb{P}(E)\mathbb{P}(F^c) \end{aligned}$$

$\therefore\ E \ \&\ F^c$ are independent □

Independence of Three Events

Definition

Let $E_1, E_2, E_3 \subseteq \Omega$ be three events from sample space Ω of an experiment.

Then E_1, E_2, E_3 are **(mutually) independent** if the following are all true:

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2)$$
$$\mathbb{P}(E_1 \cap E_3) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_3)$$
$$\mathbb{P}(E_2 \cap E_3) = \mathbb{P}(E_2) \cdot \mathbb{P}(E_3)$$
$$\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2) \cdot \mathbb{P}(E_3)$$

Note that any of the mutually independent events E_1, E_2, E_3 can be replaced by their corresponding complements and still remain mutually independent:

$$\mathbb{P}(E_1 \cap E_2^c) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2^c)$$
$$\mathbb{P}(E_1^c \cap E_2) = \mathbb{P}(E_1^c) \cdot \mathbb{P}(E_2)$$
$$\mathbb{P}(E_1^c \cap E_2^c) = \mathbb{P}(E_1^c) \cdot \mathbb{P}(E_2^c)$$
$$\mathbb{P}(E_1 \cap E_2^c \cap E_3) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2^c) \cdot \mathbb{P}(E_3)$$
$$\mathbb{P}(E_1^c \cap E_2 \cap E_3^c) = \mathbb{P}(E_1^c) \cdot \mathbb{P}(E_2) \cdot \mathbb{P}(E_3^c)$$

etc...

Independence of Four Events

Definition

Let $E_1, E_2, E_3, E_4 \subseteq \Omega$ be four events from sample space Ω of an experiment.

Then E_1, E_2, E_3, E_4 are **(mutually) independent** if the following are all true:

$$\begin{aligned}
\mathbb{P}(E_1 \cap E_2) &= \mathbb{P}(E_1) \cdot \mathbb{P}(E_2) \\
\mathbb{P}(E_1 \cap E_3) &= \mathbb{P}(E_1) \cdot \mathbb{P}(E_3) \\
\mathbb{P}(E_1 \cap E_4) &= \mathbb{P}(E_1) \cdot \mathbb{P}(E_4) \\
\mathbb{P}(E_2 \cap E_3) &= \mathbb{P}(E_2) \cdot \mathbb{P}(E_3) \\
\mathbb{P}(E_2 \cap E_4) &= \mathbb{P}(E_2) \cdot \mathbb{P}(E_4) \\
\mathbb{P}(E_3 \cap E_4) &= \mathbb{P}(E_3) \cdot \mathbb{P}(E_4)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(E_1 \cap E_2 \cap E_3) &= \mathbb{P}(E_1) \cdot \mathbb{P}(E_2) \cdot \mathbb{P}(E_3) \\
\mathbb{P}(E_1 \cap E_2 \cap E_4) &= \mathbb{P}(E_1) \cdot \mathbb{P}(E_2) \cdot \mathbb{P}(E_4) \\
\mathbb{P}(E_1 \cap E_3 \cap E_4) &= \mathbb{P}(E_1) \cdot \mathbb{P}(E_3) \cdot \mathbb{P}(E_4) \\
\mathbb{P}(E_2 \cap E_3 \cap E_4) &= \mathbb{P}(E_2) \cdot \mathbb{P}(E_3) \cdot \mathbb{P}(E_4)
\end{aligned}$$

$$\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2) \cdot \mathbb{P}(E_3) \cdot \mathbb{P}(E_4)$$

Textbook Logistics for Section 2.5

- Difference(s) in Terminology:

TEXTBOOK TERMINOLOGY	SLIDES/OUTLINE TERMINOLOGY
Null Event $\emptyset$	Empty Set $\emptyset$
Number of Outcomes in E	Measure of E

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Sample Space	$\mathcal{S}$	Ω
Complement of Event	A'	A^c
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Measure of Event	$N(A)$	$\|A\|$
k-Permutations of n-element Set	$P_{k,n}$	P_k^n
Conditional Probability	$P(A\|B)$	$\mathbb{P}(A\|B)$

Fin

Fin.