Discrete r.v.'s: pmf's & cdf's

Engineering Statistics
Section 3.2

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Probability Mass Function (pmf) of a Discrete r.v.

The **pmf** assigns a probability to each possible value of a discrete r.v.:

Definition

(pmf of a Discrete Random Variable)

Let X be a **discrete** random variable.
Then, its **pmf**, denoted as $p_X(k)$, is defined as follows:

$$p_X(k) := \mathbb{P}(X = k) \ \ \forall k \in \text{Supp}(X)$$

Corollary

(pmf Axioms)

*Let X be a **discrete** random variable. Then, its **pmf** $p_X(k)$ satisfies*

Non-negativity on its Support: $$p_X(k) \geq 0 \qquad \forall k \in \textit{Supp}(X)$$

Universal Sum of Unity: $$\sum_{k \in \textit{Supp}(X)} p_X(k) = 1$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$$\begin{array}{llll} W(AAA)=1 & X(AAA)=1 & Y(AAA)=3 & Z(AAA)=3-0 = 3 \\ W(AAF)=0 & X(AAF)=1 & Y(AAF)=2 & Z(AAF)=2-1 = 1 \\ W(AFA)=1 & X(AFA)=1 & Y(AFA)=2 & Z(AFA)=2-1 = 1 \\ W(AFF)=0 & X(AFF)=1 & Y(AFF)=1 & Z(AFF)=1-2 = -1 \\ W(FAA)=1 & X(FAA)=1 & Y(FAA)=2 & Z(FAA)=2-1 = 1 \\ W(FAF)=0 & X(FAF)=1 & Y(FAF)=1 & Z(FAF)=1-2 = -1 \\ W(FFA)=1 & X(FFA)=1 & Y(FFA)=1 & Z(FFA)=1-2 = -1 \\ W(FFF)=0 & X(FFF)=0 & Y(FFF)=0 & Z(FFF)=0-3 = -3 \end{array}$$

Supp$(W) = \{0, 1\} \implies$ Supp(W) is countable $\implies W$ is discrete
Supp$(X) = \{0, 1\} \implies$ Supp(X) is countable $\implies X$ is discrete
Supp$(Y) = \{0, 1, 2, 3\} \implies$ Supp(Y) is countable $\implies Y$ is discrete
Supp$(Z) = \{-3, -1, 1, 3\} \implies$ Supp(Z) is countable $\implies Z$ is discrete

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables

$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$W(AAA) = 1$	$X(AAA) = 1$	$Y(AAA) = 3$	$Z(AAA) = 3 - 0 = 3$
W(AAF) = 0	$X(AAF) = 1$	$Y(AAF) = 2$	$Z(AAF) = 2 - 1 = 1$
$W(AFA) = 1$	$X(AFA) = 1$	$Y(AFA) = 2$	$Z(AFA) = 2 - 1 = 1$
W(AFF) = 0	$X(AFF) = 1$	$Y(AFF) = 1$	$Z(AFF) = 1 - 2 = -1$
$W(FAA) = 1$	$X(FAA) = 1$	$Y(FAA) = 2$	$Z(FAA) = 2 - 1 = 1$
W(FAF) = 0	$X(FAF) = 1$	$Y(FAF) = 1$	$Z(FAF) = 1 - 2 = -1$
$W(FFA) = 1$	$X(FFA) = 1$	$Y(FFA) = 1$	$Z(FFA) = 1 - 2 = -1$
W(FFF) = 0	$X(FFF) = 0$	$Y(FFF) = 0$	$Z(FFF) = 0 - 3 = -3$

$$p_W(\mathbf{0}) = \mathbb{P}(\mathbf{W = 0}) = \mathbb{P}(\omega \in \{\mathbf{AAF, AFF, FAF, FFF}\}) = \frac{4}{8} = \frac{1}{2}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables

$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

W(AAA) = 1	$X(AAA) = 1$	$Y(AAA) = 3$	$Z(AAA) = 3 - 0 = 3$
$W(AAF) = 0$	$X(AAF) = 1$	$Y(AAF) = 2$	$Z(AAF) = 2 - 1 = 1$
W(AFA) = 1	$X(AFA) = 1$	$Y(AFA) = 2$	$Z(AFA) = 2 - 1 = 1$
$W(AFF) = 0$	$X(AFF) = 1$	$Y(AFF) = 1$	$Z(AFF) = 1 - 2 = -1$
W(FAA) = 1	$X(FAA) = 1$	$Y(FAA) = 2$	$Z(FAA) = 2 - 1 = 1$
$W(FAF) = 0$	$X(FAF) = 1$	$Y(FAF) = 1$	$Z(FAF) = 1 - 2 = -1$
W(FFA) = 1	$X(FFA) = 1$	$Y(FFA) = 1$	$Z(FFA) = 1 - 2 = -1$
$W(FFF) = 0$	$X(FFF) = 0$	$Y(FFF) = 0$	$Z(FFF) = 0 - 3 = -3$

$$p_W(\mathbf{1}) = \mathbb{P}(\mathbf{W = 1}) = \mathbb{P}(\omega \in \{\mathbf{AAA, AFA, FAA, FFA}\}) = \frac{4}{8} = \frac{1}{2}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)

Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables

$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)

$X \equiv$ If car has any available seats (1 = Yes, 0 = No)

$Y \equiv$ Number of available seats in car

$Z \equiv$ Difference in # of available and occupied seats

Then:

$$\begin{array}{llll}
W(AAA)=1 & X(AAA)=1 & Y(AAA)=3 & Z(AAA)=3-0 = 3 \\
W(AAF)=0 & X(AAF)=1 & Y(AAF)=2 & Z(AAF)=2-1 = 1 \\
W(AFA)=1 & X(AFA)=1 & Y(AFA)=2 & Z(AFA)=2-1 = 1 \\
W(AFF)=0 & X(AFF)=1 & Y(AFF)=1 & Z(AFF)=1-2 = -1 \\
W(FAA)=1 & X(FAA)=1 & Y(FAA)=2 & Z(FAA)=2-1 = 1 \\
W(FAF)=0 & X(FAF)=1 & Y(FAF)=1 & Z(FAF)=1-2 = -1 \\
W(FFA)=1 & X(FFA)=1 & Y(FFA)=1 & Z(FFA)=1-2 = -1 \\
W(FFF)=0 & X(FFF)=0 & Y(FFF)=0 & Z(FFF)=0-3 = -3
\end{array}$$

$$\therefore \quad p_W(k) = \begin{cases} 1/2 & \text{, if } k=0 \\ 1/2 & \text{, if } k=1 \end{cases}$$

OR

k	0	1
$p_W(k)$	$1/2$	$1/2$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)

Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables

$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)

$X \equiv$ If car has any available seats (1 = Yes, 0 = No)

$Y \equiv$ Number of available seats in car

$Z \equiv$ Difference in # of available and occupied seats

Then:

$W(AAA) = 1$	$X(AAA) = 1$	$Y(AAA) = 3$	$Z(AAA) = 3 - 0 = 3$
$W(AAF) = 0$	$X(AAF) = 1$	$Y(AAF) = 2$	$Z(AAF) = 2 - 1 = 1$
$W(AFA) = 1$	$X(AFA) = 1$	$Y(AFA) = 2$	$Z(AFA) = 2 - 1 = 1$
$W(AFF) = 0$	$X(AFF) = 1$	$Y(AFF) = 1$	$Z(AFF) = 1 - 2 = -1$
$W(FAA) = 1$	$X(FAA) = 1$	$Y(FAA) = 2$	$Z(FAA) = 2 - 1 = 1$
$W(FAF) = 0$	$X(FAF) = 1$	$Y(FAF) = 1$	$Z(FAF) = 1 - 2 = -1$
$W(FFA) = 1$	$X(FFA) = 1$	$Y(FFA) = 1$	$Z(FFA) = 1 - 2 = -1$
$W(FFF) = 0$	**X(FFF) = 0**	$Y(FFF) = 0$	$Z(FFF) = 0 - 3 = -3$

$$p_X(\mathbf{0}) = \mathbb{P}(\mathbf{X = 0}) = \mathbb{P}(\omega \in \{\mathbf{FFF}\}) = \frac{|\{FFF\}|}{|\Omega|} = \frac{1}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$$
\begin{array}{llll}
W(AAA) = 1 & \mathbf{X(AAA) = 1} & Y(AAA) = 3 & Z(AAA) = 3 - 0 = 3 \\
W(AAF) = 0 & \mathbf{X(AAF) = 1} & Y(AAF) = 2 & Z(AAF) = 2 - 1 = 1 \\
W(AFA) = 1 & \mathbf{X(AFA) = 1} & Y(AFA) = 2 & Z(AFA) = 2 - 1 = 1 \\
W(AFF) = 0 & \mathbf{X(AFF) = 1} & Y(AFF) = 1 & Z(AFF) = 1 - 2 = -1 \\
W(FAA) = 1 & \mathbf{X(FAA) = 1} & Y(FAA) = 2 & Z(FAA) = 2 - 1 = 1 \\
W(FAF) = 0 & \mathbf{X(FAF) = 1} & Y(FAF) = 1 & Z(FAF) = 1 - 2 = -1 \\
W(FFA) = 1 & \mathbf{X(FFA) = 1} & Y(FFA) = 1 & Z(FFA) = 1 - 2 = -1 \\
W(FFF) = 0 & X(FFF) = 0 & Y(FFF) = 0 & Z(FFF) = 0 - 3 = -3
\end{array}
$$

$$p_X(\mathbf{1}) = \mathbb{P}(\mathbf{X = 1}) = \mathbb{P}(\omega \in \{\mathbf{AAA,AAF,AFA,AFF,FAA,FAF,FFA}\}) = \frac{7}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$$\begin{array}{llll}
W(AAA)=1 & X(AAA)=1 & Y(AAA)=3 & Z(AAA)=3-0 = 3 \\
W(AAF)=0 & X(AAF)=1 & Y(AAF)=2 & Z(AAF)=2-1 = 1 \\
W(AFA)=1 & X(AFA)=1 & Y(AFA)=2 & Z(AFA)=2-1 = 1 \\
W(AFF)=0 & X(AFF)=1 & Y(AFF)=1 & Z(AFF)=1-2 = -1 \\
W(FAA)=1 & X(FAA)=1 & Y(FAA)=2 & Z(FAA)=2-1 = 1 \\
W(FAF)=0 & X(FAF)=1 & Y(FAF)=1 & Z(FAF)=1-2 = -1 \\
W(FFA)=1 & X(FFA)=1 & Y(FFA)=1 & Z(FFA)=1-2 = -1 \\
W(FFF)=0 & X(FFF)=0 & Y(FFF)=0 & Z(FFF)=0-3 = -3
\end{array}$$

$$\therefore \quad p_X(k) = \begin{cases} 1/8 & \text{, if } k=0 \\ 7/8 & \text{, if } k=1 \end{cases}$$

OR

k	0	1
$p_X(k)$	$1/8$	$7/8$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$W(AAA) = 1$	$X(AAA) = 1$	$Y(AAA) = 3$	$Z(AAA) = 3 - 0 = 3$
$W(AAF) = 0$	$X(AAF) = 1$	$Y(AAF) = 2$	$Z(AAF) = 2 - 1 = 1$
$W(AFA) = 1$	$X(AFA) = 1$	$Y(AFA) = 2$	$Z(AFA) = 2 - 1 = 1$
$W(AFF) = 0$	$X(AFF) = 1$	$Y(AFF) = 1$	$Z(AFF) = 1 - 2 = -1$
$W(FAA) = 1$	$X(FAA) = 1$	$Y(FAA) = 2$	$Z(FAA) = 2 - 1 = 1$
$W(FAF) = 0$	$X(FAF) = 1$	$Y(FAF) = 1$	$Z(FAF) = 1 - 2 = -1$
$W(FFA) = 1$	$X(FFA) = 1$	$Y(FFA) = 1$	$Z(FFA) = 1 - 2 = -1$
$W(FFF) = 0$	$X(FFF) = 0$	**Y(FFF) = 0**	$Z(FFF) = 0 - 3 = -3$

$$p_Y(\mathbf{0}) = \mathbb{P}(\mathbf{Y = 0}) = \mathbb{P}(\omega \in \{\mathbf{FFF}\}) = \frac{|\{FFF\}|}{|\Omega|} = \frac{1}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables

$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$$\begin{array}{llll}
W(AAA)=1 & X(AAA)=1 & Y(AAA)=3 & Z(AAA)=3-0 = 3 \\
W(AAF)=0 & X(AAF)=1 & Y(AAF)=2 & Z(AAF)=2-1 = 1 \\
W(AFA)=1 & X(AFA)=1 & Y(AFA)=2 & Z(AFA)=2-1 = 1 \\
W(AFF)=0 & X(AFF)=1 & \mathbf{Y(AFF) = 1} & Z(AFF)=1-2 = -1 \\
W(FAA)=1 & X(FAA)=1 & Y(FAA)=2 & Z(FAA)=2-1 = 1 \\
W(FAF)=0 & X(FAF)=1 & \mathbf{Y(FAF) = 1} & Z(FAF)=1-2 = -1 \\
W(FFA)=1 & X(FFA)=1 & \mathbf{Y(FFA) = 1} & Z(FFA)=1-2 = -1 \\
W(FFF)=0 & X(FFF)=0 & Y(FFF)=0 & Z(FFF)=0-3 = -3
\end{array}$$

$$p_Y(\mathbf{1}) = \mathbb{P}(\mathbf{Y = 1}) = \mathbb{P}(\omega \in \{\mathbf{AFF,FAF,FFA}\}) = \frac{3}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$W(AAA) = 1$	$X(AAA) = 1$	$Y(AAA) = 3$	$Z(AAA) = 3 - 0 = 3$
$W(AAF) = 0$	$X(AAF) = 1$	**Y(AAF) = 2**	$Z(AAF) = 2 - 1 = 1$
$W(AFA) = 1$	$X(AFA) = 1$	**Y(AFA) = 2**	$Z(AFA) = 2 - 1 = 1$
$W(AFF) = 0$	$X(AFF) = 1$	$Y(AFF) = 1$	$Z(AFF) = 1 - 2 = -1$
$W(FAA) = 1$	$X(FAA) = 1$	**Y(FAA) = 2**	$Z(FAA) = 2 - 1 = 1$
$W(FAF) = 0$	$X(FAF) = 1$	$Y(FAF) = 1$	$Z(FAF) = 1 - 2 = -1$
$W(FFA) = 1$	$X(FFA) = 1$	$Y(FFA) = 1$	$Z(FFA) = 1 - 2 = -1$
$W(FFF) = 0$	$X(FFF) = 0$	$Y(FFF) = 0$	$Z(FFF) = 0 - 3 = -3$

$$p_Y(\mathbf{2}) = \mathbb{P}(\mathbf{Y = 2}) = \mathbb{P}(\omega \in \{\mathbf{AAF,AFA,FAA}\}) = \frac{3}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$W(AAA) = 1$	$X(AAA) = 1$	**Y(AAA) = 3**	$Z(AAA) = 3 - 0 = 3$
$W(AAF) = 0$	$X(AAF) = 1$	$Y(AAF) = 2$	$Z(AAF) = 2 - 1 = 1$
$W(AFA) = 1$	$X(AFA) = 1$	$Y(AFA) = 2$	$Z(AFA) = 2 - 1 = 1$
$W(AFF) = 0$	$X(AFF) = 1$	$Y(AFF) = 1$	$Z(AFF) = 1 - 2 = -1$
$W(FAA) = 1$	$X(FAA) = 1$	$Y(FAA) = 2$	$Z(FAA) = 2 - 1 = 1$
$W(FAF) = 0$	$X(FAF) = 1$	$Y(FAF) = 1$	$Z(FAF) = 1 - 2 = -1$
$W(FFA) = 1$	$X(FFA) = 1$	$Y(FFA) = 1$	$Z(FFA) = 1 - 2 = -1$
$W(FFF) = 0$	$X(FFF) = 0$	$Y(FFF) = 0$	$Z(FFF) = 0 - 3 = -3$

$$p_Y(\mathbf{3}) = \mathbb{P}(\mathbf{Y = 3}) = \mathbb{P}(\omega \in \{\mathbf{AAA}\}) = \frac{|\{AAA\}|}{|\Omega|} = \frac{1}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$$\begin{array}{llll}
W(AAA)=1 & X(AAA)=1 & Y(AAA)=3 & Z(AAA)=3-0 = 3 \\
W(AAF)=0 & X(AAF)=1 & Y(AAF)=2 & Z(AAF)=2-1 = 1 \\
W(AFA)=1 & X(AFA)=1 & Y(AFA)=2 & Z(AFA)=2-1 = 1 \\
W(AFF)=0 & X(AFF)=1 & Y(AFF)=1 & Z(AFF)=1-2 = -1 \\
W(FAA)=1 & X(FAA)=1 & Y(FAA)=2 & Z(FAA)=2-1 = 1 \\
W(FAF)=0 & X(FAF)=1 & Y(FAF)=1 & Z(FAF)=1-2 = -1 \\
W(FFA)=1 & X(FFA)=1 & Y(FFA)=1 & Z(FFA)=1-2 = -1 \\
W(FFF)=0 & X(FFF)=0 & Y(FFF)=0 & Z(FFF)=0-3 = -3
\end{array}$$

$$\therefore \quad p_Y(k) = \begin{cases} 1/8 & \text{, if } k=0 \\ 3/8 & \text{, if } k=1 \\ 3/8 & \text{, if } k=2 \\ 1/8 & \text{, if } k=3 \end{cases}$$

OR

k	0	1	2	3
$p_Y(k)$	1/8	3/8	3/8	1/8

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$W(AAA) = 1$	$X(AAA) = 1$	$Y(AAA) = 3$	$Z(AAA) = 3 - 0 = 3$
$W(AAF) = 0$	$X(AAF) = 1$	$Y(AAF) = 2$	$Z(AAF) = 2 - 1 = 1$
$W(AFA) = 1$	$X(AFA) = 1$	$Y(AFA) = 2$	$Z(AFA) = 2 - 1 = 1$
$W(AFF) = 0$	$X(AFF) = 1$	$Y(AFF) = 1$	$Z(AFF) = 1 - 2 = -1$
$W(FAA) = 1$	$X(FAA) = 1$	$Y(FAA) = 2$	$Z(FAA) = 2 - 1 = 1$
$W(FAF) = 0$	$X(FAF) = 1$	$Y(FAF) = 1$	$Z(FAF) = 1 - 2 = -1$
$W(FFA) = 1$	$X(FFA) = 1$	$Y(FFA) = 1$	$Z(FFA) = 1 - 2 = -1$
$W(FFF) = 0$	$X(FFF) = 0$	$Y(FFF) = 0$	**Z(FFF)** $= 0 - 3 =$ **-3**

$$p_Z(\mathbf{-3}) = \mathbb{P}(\mathbf{Z = -3}) = \mathbb{P}(\omega \in \{\mathbf{FFF}\}) = \frac{|\{FFF\}|}{|\Omega|} = \frac{1}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$W(AAA) = 1$	$X(AAA) = 1$	$Y(AAA) = 3$	$Z(AAA) = 3 - 0$	$=$	3
$W(AAF) = 0$	$X(AAF) = 1$	$Y(AAF) = 2$	$Z(AAF) = 2 - 1$	$=$	1
$W(AFA) = 1$	$X(AFA) = 1$	$Y(AFA) = 2$	$Z(AFA) = 2 - 1$	$=$	1
$W(AFF) = 0$	$X(AFF) = 1$	$Y(AFF) = 1$	**Z(AFF)** $= 1 - 2$	$=$	**-1**
$W(FAA) = 1$	$X(FAA) = 1$	$Y(FAA) = 2$	$Z(FAA) = 2 - 1$	$=$	1
$W(FAF) = 0$	$X(FAF) = 1$	$Y(FAF) = 1$	**Z(FAF)** $= 1 - 2$	$=$	**-1**
$W(FFA) = 1$	$X(FFA) = 1$	$Y(FFA) = 1$	**Z(FFA)** $= 1 - 2$	$=$	**-1**
$W(FFF) = 0$	$X(FFF) = 0$	$Y(FFF) = 0$	$Z(FFF) = 0 - 3$	$=$	-3

$$p_Z(\mathbf{-1}) = \mathbb{P}(\mathbf{Z = -1}) = \mathbb{P}(\omega \in \{\mathbf{AFF,FAF,FFA}\}) = \frac{3}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$$\begin{array}{llll} W(AAA)=1 & X(AAA)=1 & Y(AAA)=3 & Z(AAA)=3-0 = 3 \\ W(AAF)=0 & X(AAF)=1 & Y(AAF)=2 & \mathbf{Z(AAF)}=2-1 = \mathbf{1} \\ W(AFA)=1 & X(AFA)=1 & Y(AFA)=2 & \mathbf{Z(AFA)}=2-1 = \mathbf{1} \\ W(AFF)=0 & X(AFF)=1 & Y(AFF)=1 & Z(AFF)=1-2 = -1 \\ W(FAA)=1 & X(FAA)=1 & Y(FAA)=2 & \mathbf{Z(FAA)}=2-1 = \mathbf{1} \\ W(FAF)=0 & X(FAF)=1 & Y(FAF)=1 & Z(FAF)=1-2 = -1 \\ W(FFA)=1 & X(FFA)=1 & Y(FFA)=1 & Z(FFA)=1-2 = -1 \\ W(FFF)=0 & X(FFF)=0 & Y(FFF)=0 & Z(FFF)=0-3 = -3 \end{array}$$

$$p_Z(\mathbf{1}) = \mathbb{P}(\mathbf{Z = 1}) = \mathbb{P}(\omega \in \{\mathbf{AAF,AFA,FAA}\}) = \frac{3}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)

Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables

$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)

$X \equiv$ If car has any available seats (1 = Yes, 0 = No)

$Y \equiv$ Number of available seats in car

$Z \equiv$ Difference in # of available and occupied seats

Then:

$$\begin{array}{llll}
W(AAA) = 1 & X(AAA) = 1 & Y(AAA) = 3 & \mathbf{Z(AAA)} = 3 - 0 = \mathbf{3} \\
W(AAF) = 0 & X(AAF) = 1 & Y(AAF) = 2 & Z(AAF) = 2 - 1 = 1 \\
W(AFA) = 1 & X(AFA) = 1 & Y(AFA) = 2 & Z(AFA) = 2 - 1 = 1 \\
W(AFF) = 0 & X(AFF) = 1 & Y(AFF) = 1 & Z(AFF) = 1 - 2 = -1 \\
W(FAA) = 1 & X(FAA) = 1 & Y(FAA) = 2 & Z(FAA) = 2 - 1 = 1 \\
W(FAF) = 0 & X(FAF) = 1 & Y(FAF) = 1 & Z(FAF) = 1 - 2 = -1 \\
W(FFA) = 1 & X(FFA) = 1 & Y(FFA) = 1 & Z(FFA) = 1 - 2 = -1 \\
W(FFF) = 0 & X(FFF) = 0 & Y(FFF) = 0 & Z(FFF) = 0 - 3 = -3
\end{array}$$

$$p_Z(\mathbf{3}) = \mathbb{P}(\mathbf{Z = 3}) = \mathbb{P}(\omega \in \{\mathbf{AAA}\}) = \frac{|\{AAA\}|}{|\Omega|} = \frac{1}{8}$$

pmf of a Discrete Random Variable (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$$
\begin{array}{llll}
W(AAA) = 1 & X(AAA) = 1 & Y(AAA) = 3 & Z(AAA) = 3 - 0 = 3 \\
W(AAF) = 0 & X(AAF) = 1 & Y(AAF) = 2 & Z(AAF) = 2 - 1 = 1 \\
W(AFA) = 1 & X(AFA) = 1 & Y(AFA) = 2 & Z(AFA) = 2 - 1 = 1 \\
W(AFF) = 0 & X(AFF) = 1 & Y(AFF) = 1 & Z(AFF) = 1 - 2 = -1 \\
W(FAA) = 1 & X(FAA) = 1 & Y(FAA) = 2 & Z(FAA) = 2 - 1 = 1 \\
W(FAF) = 0 & X(FAF) = 1 & Y(FAF) = 1 & Z(FAF) = 1 - 2 = -1 \\
W(FFA) = 1 & X(FFA) = 1 & Y(FFA) = 1 & Z(FFA) = 1 - 2 = -1 \\
W(FFF) = 0 & X(FFF) = 0 & Y(FFF) = 0 & Z(FFF) = 0 - 3 = -3
\end{array}
$$

$$
\therefore \quad p_Z(k) = \begin{cases} 1/8 & \text{, if } k = -3 \\ 3/8 & \text{, if } k = -1 \\ 3/8 & \text{, if } k = 1 \\ 1/8 & \text{, if } k = 3 \end{cases}
$$

OR

k	-3	-1	1	3
$p_Z(k)$	$1/8$	$3/8$	$3/8$	$1/8$

Verification of pmf's

k	0	1
$p_W(k)$	$1/2$	$1/2$

$$\sum_{k \in \text{Supp}(W)} p_W(k) = p_W(0) + p_W(1) = \frac{1}{2} + \frac{1}{2} = 1 \quad \checkmark$$

---

k	0	1
$p_X(k)$	$1/8$	$7/8$

$$\sum_{k \in \text{Supp}(X)} p_X(k) = p_X(0) + p_X(1) = \frac{1}{8} + \frac{7}{8} = 1 \quad \checkmark$$

---

k	0	1	2	3
$p_Y(k)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\sum_{k \in \text{Supp}(Y)} p_Y(k) = p_Y(0) + p_Y(1) + p_Y(2) + p_Y(3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1 \quad \checkmark$$

---

k	-3	-1	1	3
$p_Z(k)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\sum_{k \in \text{Supp}(Z)} p_Z(k) = p_Z(-3) + p_Z(-1) + p_Z(1) + p_Z(3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1 \quad \checkmark$$

Cumulative Density Function (cdf) of a Discrete r.v.

Definition

(cdf of a Discrete Random Variable)

Let X be a **discrete** random variable s.t. Supp$(X) = \{k_1, k_2, k_3, \cdots\}$
Then, its **cdf**, denoted as $F_X(x)$, is defined as follows:

$$F_X(x) := \mathbb{P}(X \leq x) = \sum_{k_i \leq x} p_X(k_i) \quad \forall x \in \mathbb{R}$$

Corollary

(cdf Axioms)

*Let X be a **discrete** random variable. Then, its **cdf** $F_X(x)$, satisfies*

Eventually Zero (One) to the Left (Right): $\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to +\infty} F_X(x) = 1$

Non-decreasing: $x_1 \leq x_2 \implies F_X(x_1) \leq F_X(x_2)$

Right-continuous: $\lim_{x \downarrow x_0} F_X(x) = F_X(x_0) \quad \forall x_0 \in \mathbb{R}$

Piecewise Constant: (*AKA step function*)

Computing cdf's from pmf's

$$p_W(k) = \begin{cases} 1/2 & \text{, if } k = 0 \\ 1/2 & \text{, if } k = 1 \end{cases} \implies F_W(x) = \begin{cases} 0 & \text{, if } x < 0 \\ \frac{1}{2} & \text{, if } 0 \le x < 1 \\ \frac{1}{2} + \frac{1}{2} & \text{, if } 1 \le x \end{cases}$$

$$p_X(k) = \begin{cases} 1/8 & \text{, if } k = 0 \\ 7/8 & \text{, if } k = 1 \end{cases} \implies F_X(x) = \begin{cases} 0 & \text{, if } x < 0 \\ \frac{1}{8} & \text{, if } 0 \le x < 1 \\ \frac{1}{8} + \frac{7}{8} & \text{, if } 1 \le x \end{cases}$$

$$p_Y(k) = \begin{cases} 1/8 & \text{, if } k = 0 \\ 3/8 & \text{, if } k = 1 \\ 3/8 & \text{, if } k = 2 \\ 1/8 & \text{, if } k = 3 \end{cases} \implies F_Y(x) = \begin{cases} 0 & \text{, if } x < 0 \\ \frac{1}{8} & \text{, if } 0 \le x < 1 \\ \frac{1}{8} + \frac{3}{8} & \text{, if } 1 \le x < 2 \\ \frac{1}{8} + \frac{3}{8} + \frac{3}{8} & \text{, if } 2 \le x < 3 \\ \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} & \text{, if } 3 \le x \end{cases}$$

$$p_Z(k) = \begin{cases} 1/8 & \text{, if } k = -3 \\ 3/8 & \text{, if } k = -1 \\ 3/8 & \text{, if } k = 1 \\ 1/8 & \text{, if } k = 3 \end{cases} \implies F_Z(x) = \begin{cases} 0 & \text{, if } x < -3 \\ \frac{1}{8} & \text{, if } -3 \le x < -1 \\ \frac{1}{8} + \frac{3}{8} & \text{, if } -1 \le x < 1 \\ \frac{1}{8} + \frac{3}{8} + \frac{3}{8} & \text{, if } 1 \le x < 3 \\ 1 & \text{, if } 3 \le x \end{cases}$$

Computing cdf's from pmf's

$$p_W(k) = \begin{cases} 1/2 & \text{, if } k = 0 \\ 1/2 & \text{, if } k = 1 \end{cases} \implies F_W(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1/2 & \text{, if } 0 \leq x < 1 \\ 1 & \text{, if } 1 \leq x \end{cases}$$

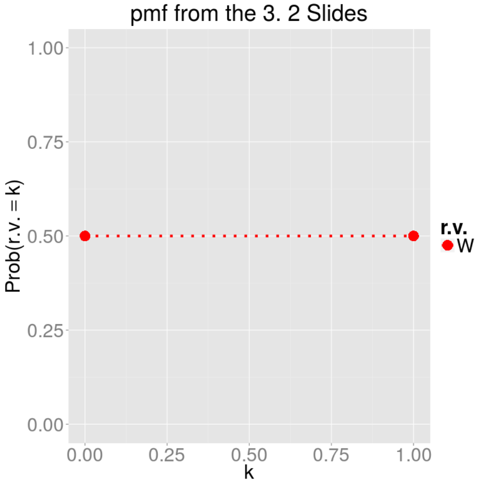
$$p_X(k) = \begin{cases} 1/8 & \text{, if } k = 0 \\ 7/8 & \text{, if } k = 1 \end{cases} \implies F_X(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1/8 & \text{, if } 0 \leq x < 1 \\ 1 & \text{, if } 1 \leq x \end{cases}$$

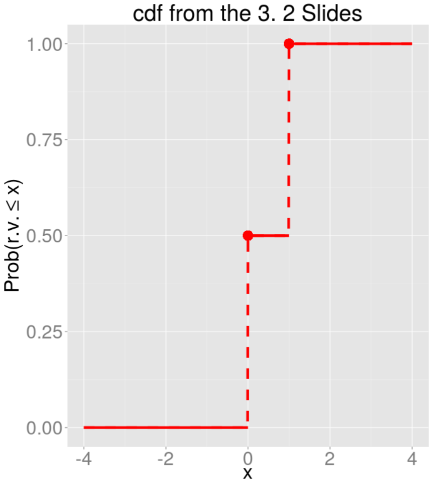
$$p_Y(k) = \begin{cases} 1/8 & \text{, if } k = 0 \\ 3/8 & \text{, if } k = 1 \\ 3/8 & \text{, if } k = 2 \\ 1/8 & \text{, if } k = 3 \end{cases} \implies F_Y(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1/8 & \text{, if } 0 \leq x < 1 \\ 1/2 & \text{, if } 1 \leq x < 2 \\ 7/8 & \text{, if } 2 \leq x < 3 \\ 1 & \text{, if } 3 \leq x \end{cases}$$

$$p_Z(k) = \begin{cases} 1/8 & \text{, if } k = -3 \\ 3/8 & \text{, if } k = -1 \\ 3/8 & \text{, if } k = 1 \\ 1/8 & \text{, if } k = 3 \end{cases} \implies F_Z(x) = \begin{cases} 0 & \text{, if } x < -3 \\ 1/8 & \text{, if } -3 \leq x < -1 \\ 1/2 & \text{, if } -1 \leq x < 1 \\ 7/8 & \text{, if } 1 \leq x < 3 \\ 1 & \text{, if } 3 \leq x \end{cases}$$

Plots of pmf & cdf of each Discrete Random Variable

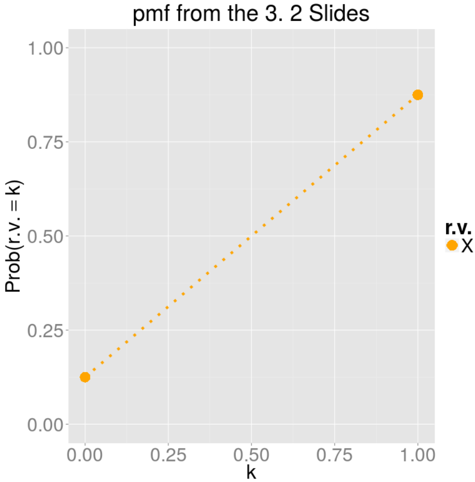
k	0	1
$p_W(k)$	$1/2$	$1/2$

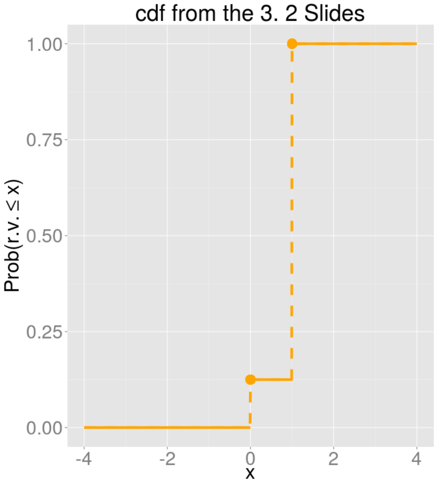




Plots of pmf & cdf of each Discrete Random Variable

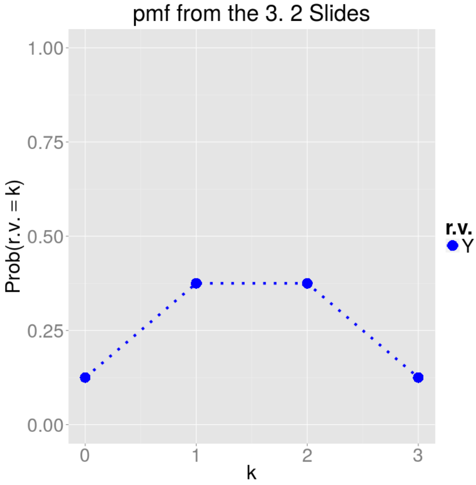
k	0	1
$p_X(k)$	$1/8$	$7/8$

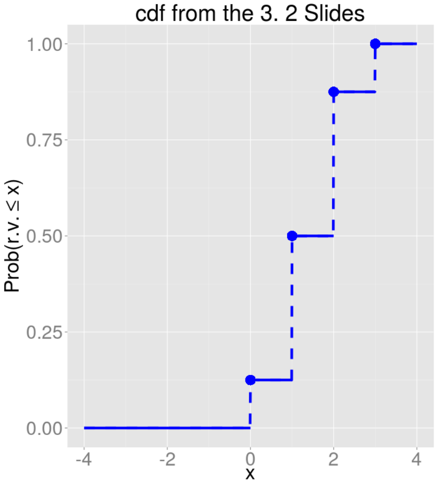




Plots of pmf & cdf of each Discrete Random Variable

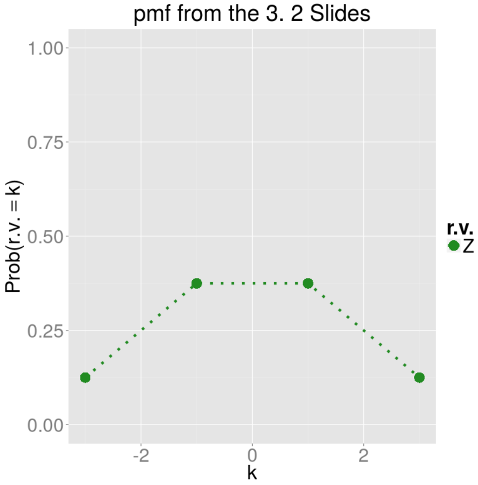
k	0	1	2	3
$p_Y(k)$	$1/8$	$3/8$	$3/8$	$1/8$

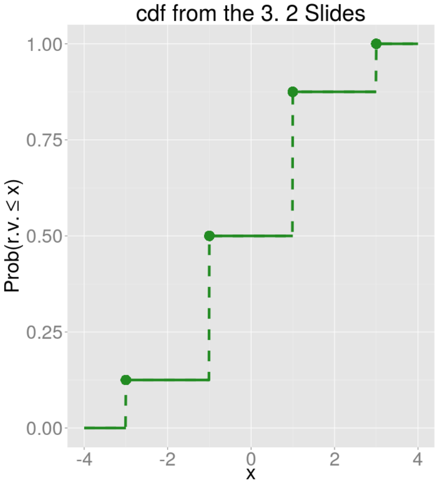




Plots of pmf & cdf of each Discrete Random Variable

k	-3	-1	1	3
$p_Z(k)$	$1/8$	$3/8$	$3/8$	$1/8$





Computing Probabilities using a Discrete cdf

Theorem

*Let X be a **discrete** r.v. with cdf $F_X(x)$. Let scalars $a, b \in \mathbb{R}$ s.t. $a < b$. Then:*

$$\begin{aligned}
\mathbb{P}(X \leq a) &= F_X(a) \\
\mathbb{P}(X < a) &= F_X(a-)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(a \leq X \leq b) &= F_X(b) - F_X(a-) \\
\mathbb{P}(a < X < b) &= F_X(b-) - F_X(a) \\
\mathbb{P}(a < X \leq b) &= F_X(b) - F_X(a) \\
\mathbb{P}(a \leq X < b) &= F_X(b-) - F_X(a-)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X \geq b) &= 1 - F_X(b-) \\
\mathbb{P}(X > b) &= 1 - F_X(b)
\end{aligned}$$

$$\mathbb{P}(X = a) = F_X(a) - F_X(a-)$$

where: "$a-$" *represents the **largest** $k \in$ Supp(X) such that $k < a$*
"$b-$" *represents the **largest** $k \in$ Supp(X) such that $k < b$*

Comparison of Probabilities via pmf v.s. cdf

$$p_Y(k) = \begin{cases} 1/8 & \text{, if } k=0 \\ 3/8 & \text{, if } k=1 \\ 3/8 & \text{, if } k=2 \\ 1/8 & \text{, if } k=3 \end{cases} \implies F_Y(x) = \begin{cases} 0 & \text{, if } x<0 \\ 1/8 & \text{, if } 0 \le x < 1 \\ 1/2 & \text{, if } 1 \le x < 2 \\ 7/8 & \text{, if } 2 \le x < 3 \\ 1 & \text{, if } 3 \le x \end{cases}$$

$$\mathbb{P}(Y \le 2) = p_Y(0) + p_Y(1) + p_Y(2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

$$\mathbb{P}(Y \le 2) = F_Y(2) = \frac{7}{8}$$

$$\mathbb{P}(Y < 2) = p_Y(0) + p_Y(1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

$$\mathbb{P}(Y < 2) = F_Y(2-) = F_Y(1) = \frac{1}{2}$$

$$\mathbb{P}(1 < Y \le 3) = p_Y(2) + p_Y(3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\mathbb{P}(1 < Y \le 3) = F_Y(3) - F_Y(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

Comparison of Probabilities via pmf v.s. cdf

$$p_Y(k) = \begin{cases} 1/8 & \text{, if } k = 0 \\ 3/8 & \text{, if } k = 1 \\ 3/8 & \text{, if } k = 2 \\ 1/8 & \text{, if } k = 3 \end{cases} \implies F_Y(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1/8 & \text{, if } 0 \leq x < 1 \\ 1/2 & \text{, if } 1 \leq x < 2 \\ 7/8 & \text{, if } 2 \leq x < 3 \\ 1 & \text{, if } 3 \leq x \end{cases}$$

$$\mathbb{P}(Y \geq 2) = p_Y(2) + p_Y(3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\mathbb{P}(Y \geq 2) = 1 - F_Y(2-) = 1 - F_Y(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\mathbb{P}(Y > 2) = p_Y(3) = \frac{1}{8}$$

$$\mathbb{P}(Y > 2) = 1 - F_Y(2) = 1 - \frac{7}{8} = \frac{1}{8}$$

$$\mathbb{P}(1 \leq Y < 3) = p_Y(1) + p_Y(2) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$$

$$\mathbb{P}(1 \leq Y < 3) = F_Y(3-) - F_Y(1-) = F_Y(2) - F_Y(0) = \frac{7}{8} - \frac{1}{8} = \frac{3}{4}$$

Comparison of Probabilities via pmf v.s. cdf

$$p_Y(k) = \begin{cases} 1/8 & \text{, if } k = 0 \\ 3/8 & \text{, if } k = 1 \\ 3/8 & \text{, if } k = 2 \\ 1/8 & \text{, if } k = 3 \end{cases} \implies F_Y(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1/8 & \text{, if } 0 \leq x < 1 \\ 1/2 & \text{, if } 1 \leq x < 2 \\ 7/8 & \text{, if } 2 \leq x < 3 \\ 1 & \text{, if } 3 \leq x \end{cases}$$

$$\mathbb{P}(Y = 2) = p_Y(2) = \frac{3}{8}$$

$$\mathbb{P}(Y = 2) = F_Y(2) - F_Y(2-) = F_Y(2) - F_Y(1) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$$

$$\mathbb{P}(Y = 2.5) = p_Y(2.5) = 0 \qquad [\text{Since } 2.5 \notin \text{Supp}(Y)]$$

$$\mathbb{P}(Y = 2.5) = F_Y(2.5) - F_Y(2.5-) = F_Y(2.5) - F_Y(2) = \frac{7}{8} - \frac{7}{8} = 0$$

Textbook Logistics for Section 3.2

- Difference(s) in Terminology:

TEXTBOOK TERMINOLOGY	SLIDES/OUTLINE TERMINOLOGY
Null Event $\emptyset$	Empty Set $\emptyset$
Number of Outcomes in E	Measure of E

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Sample Space	$\mathcal{S}$	Ω
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Measure of Event	$N(A)$	$\lvert A \rvert$
Support of a r.v.	"All possible values of X"	Supp(X)
Support of a r.v.	D	Supp(X)
pmf of a r.v.	$p(x)$	$p_X(k)$
cdf of a r.v.	$F(x)$	$F_X(x)$

Textbook Logistics for Section 3.2

- Skip Parameter(s) of a Probability Distribution (pg 103)
  - Parameters are often associated with pmf's of well-known random variables
  - Parameters will be covered in Sections 3.4 & 3.6
  - Parameters also occur with continuous random variables, and hence will also be covered in Ch4.

Fin.