Discrete r.v.'s: Expected Value & Variance

Engineering Statistics
Section 3.3

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pmf's of Discrete Random Variables (Examples)

Experiment: Observe which seats in a 3-seat car are occupied (F) or not (A)
Sample Space: $\Omega = \{AAA, AAF, AFA, AFF, FAA, FAF, FFA, FFF\}$

Let random variables
$W \equiv$ If 3^{rd} seat in car is available (1 = Yes, 0 = No)
$X \equiv$ If car has any available seats (1 = Yes, 0 = No)
$Y \equiv$ Number of available seats in car
$Z \equiv$ Difference in # of available and occupied seats

Then:

$$
\begin{array}{llll}
W(AAA) = 1 & X(AAA) = 1 & Y(AAA) = 3 & Z(AAA) = 3 - 0 = 3 \\
W(AAF) = 0 & X(AAF) = 1 & Y(AAF) = 2 & Z(AAF) = 2 - 1 = 1 \\
W(AFA) = 1 & X(AFA) = 1 & Y(AFA) = 2 & Z(AFA) = 2 - 1 = 1 \\
W(AFF) = 0 & X(AFF) = 1 & Y(AFF) = 1 & Z(AFF) = 1 - 2 = -1 \\
W(FAA) = 1 & X(FAA) = 1 & Y(FAA) = 2 & Z(FAA) = 2 - 1 = 1 \\
W(FAF) = 0 & X(FAF) = 1 & Y(FAF) = 1 & Z(FAF) = 1 - 2 = -1 \\
W(FFA) = 1 & X(FFA) = 1 & Y(FFA) = 1 & Z(FFA) = 1 - 2 = -1 \\
W(FFF) = 0 & X(FFF) = 0 & Y(FFF) = 0 & Z(FFF) = 0 - 3 = -3
\end{array}
$$

Supp$(W) = \{0, 1\} \implies$ Supp(W) is countable $\implies W$ is discrete
Supp$(X) = \{0, 1\} \implies$ Supp(X) is countable $\implies X$ is discrete
Supp$(Y) = \{0, 1, 2, 3\} \implies$ Supp(Y) is countable $\implies Y$ is discrete
Supp$(Z) = \{-3, -1, 1, 3\} \implies$ Supp(Z) is countable $\implies Z$ is discrete

pmf's of Discrete Random Variables (Examples)

k	0	1
$p_W(k)$	$1/2$	$1/2$

$$\sum_{k \in \text{Supp}(W)} p_W(k) = p_W(0) + p_W(1) = \frac{1}{2} + \frac{1}{2} = 1 \ \checkmark$$

k	0	1
$p_X(k)$	$1/8$	$7/8$

$$\sum_{k \in \text{Supp}(X)} p_X(k) = p_X(0) + p_X(1) = \frac{1}{8} + \frac{7}{8} = 1 \ \checkmark$$

k	0	1	2	3
$p_Y(k)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\sum_{k \in \text{Supp}(Y)} p_Y(k) = p_Y(0) + p_Y(1) + p_Y(2) + p_Y(3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1 \ \checkmark$$

k	-3	-1	1	3
$p_Z(k)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\sum_{k \in \text{Supp}(Z)} p_Z(k) = p_Z(-3) + p_Z(-1) + p_Z(1) + p_Z(3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1 \ \checkmark$$

pmf's of Discrete r.v.'s Represent **Populations**

Recall from Chapter 1 that:

- **Inferential Statistics** draws conclusions on populations from samples.
- **Probability** draws conclusions on samples from populations.

Since the pmf of a discrete random variable indicates **probabilities**, pmf's represent **populations**, not samples!

Moreover, since probabilities can be view as **relative frequencies**, pmf's can be visualized as **histograms.**

As mentioned in Chapter 1, a population has a **mean** & a **variance.**

Since a random variable represents a population, it makes sense to talk about:

- the **mean** (more often called the **expected value**) of a random variable
- the **variance** of a random variable
- the **standard deviation** of a random variable

Expected Value (Mean) of a Discrete Random Variable

Definition

(Expected Value of a Discrete r.v.)

Let X be a **discrete** random variable with pmf $p_X(k)$.
Then the **expected value** (AKA **mean**) of X is:

$$\mathbb{E}[X] := \sum_{k \in \text{Supp}(X)} k \cdot p_X(k)$$

It's possible (but rare) that the expected value is **infinite**: $\mathbb{E}[X] = \infty$

NOTATION: The expected value of X is alternatively denoted by μ_X.

Expected Values of Discrete r.v.'s (Examples)

k	0	1
$p_W(k)$	$1/2$	$1/2$

$$\begin{aligned} \mathbb{E}[W] = \sum_{k \in \text{Supp}(W)} k \cdot p_W(k) &= 0 \cdot p_W(0) + 1 \cdot p_W(1) \\ &= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0 + \frac{1}{2} = \boxed{\frac{1}{2}} = 0.5 \end{aligned}$$

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k	0	1
$p_X(k)$	$1/8$	$7/8$

$$\begin{aligned} \mathbb{E}[X] = \sum_{k \in \text{Supp}(X)} k \cdot p_X(k) &= 0 \cdot p_X(0) + 1 \cdot p_X(1) \\ &= 0 \cdot \frac{1}{8} + 1 \cdot \frac{7}{8} = 0 + \frac{7}{8} = \boxed{\frac{7}{8}} = 0.875 \end{aligned}$$

Expected Values of Discrete r.v.'s (Examples)

k	0	1	2	3
$p_Y(k)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\begin{aligned}\mathbb{E}[Y] = \sum_{k\in\text{Supp}(Y)} k\cdot p_Y(k) &= 0\cdot p_Y(0) + 1\cdot p_Y(1) + 2\cdot p_Y(2) + 3\cdot p_Y(3) \\ &= 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} \\ &= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \boxed{\frac{3}{2}} = 1.5\end{aligned}$$

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k	-3	-1	1	3
$p_Z(k)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\begin{aligned}\mathbb{E}[Z] = \sum_{k\in\text{Supp}(Z)} k\cdot p_Z(k) &= (-3)p_Z(-3) + (-1)p_Z(-1) + 1\cdot p_Z(1) + 3\cdot p_Z(3) \\ &= (-3)\cdot\frac{1}{8} + (-1)\cdot\frac{3}{8} + 1\cdot\frac{3}{8} + 3\cdot\frac{1}{8} \\ &= -\frac{3}{8} - \frac{3}{8} + \frac{3}{8} + \frac{3}{8} = \boxed{0}\end{aligned}$$

Expected Value (Mean) of a Function of Discrete r.v.

Definition

Let X be a **discrete** random variable with pmf $p_X(k)$.
Let $h(x)$ be a single-variable function.
Then the **expected value** (AKA **mean**) of $h(X)$ is:

$$\mathbb{E}[h(X)] := \sum_{k \in \text{Supp}(X)} h(k) \cdot p_X(k)$$

It's possible (but rare) that the expected value is **infinite**: $\mathbb{E}[h(X)] = \pm\infty$

NOTATION: The expected value of $h(X)$ is alternatively denoted by $\mu_{h(X)}$.

Variance & Standard Deviation of a Discrete r.v.

Definition

(Variance & Standard Deviation of a Discrete Random Variable)

Let X be a **discrete** random variable with pmf $p_X(k)$ and mean μ_X.
Then the **variance** of X is:

$$\mathbb{V}[X] := \mathbb{E}[(X - \mu_X)^2] = \sum_{k \in \text{Supp}(X)} (k - \mu_X)^2 \cdot p_X(k)$$

Moreover, the **standard deviation** of X is: $\sigma_X := \sqrt{\mathbb{V}[X]}$

<u>NOTATION:</u> The variance of X is alternatively denoted by σ_X^2 or $\text{Var}(X)$.

Variance & Std Dev of Discrete r.v.'s (Examples)

k	0	1
$p_X(k)$	$1/8$	$7/8$

$$\mu_X = \mathbb{E}[X] = \sum_{k \in \mathsf{Supp}(X)} k \cdot p_X(k) = 0 \cdot p_X(0) + 1 \cdot p_X(1) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{7}{8} = \frac{7}{8}$$

$$\begin{aligned}
\mathbb{V}[X] &= \sum_{k \in \mathsf{Supp}(X)} (k - \mu_X)^2 \cdot p_X(k) \\
&= \left(0 - \frac{7}{8}\right)^2 \cdot p_X(0) + \left(1 - \frac{7}{8}\right)^2 \cdot p_X(1) \\
&= \frac{49}{64} \cdot \frac{1}{8} + \frac{1}{64} \cdot \frac{7}{8} = \frac{49}{(64)(8)} + \frac{7}{(64)(8)} = \frac{56}{(64)(8)} = \boxed{\frac{7}{64}}
\end{aligned}$$

$$\sigma_X = \sqrt{\mathbb{V}[X]} = \sqrt{\frac{7}{64}} = \boxed{\frac{\sqrt{7}}{8}} \approx 0.330719$$

Variance & Std Dev of Discrete r.v.'s (Examples)

k	0	1	2	3
$p_Y(k)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\begin{aligned}\mu_Y = \mathbb{E}[Y] &= \sum_{k \in \mathsf{Supp}(Y)} k \cdot p_Y(k) \\ &= 0 \cdot p_Y(0) + 1 \cdot p_Y(1) + 2 \cdot p_Y(2) + 3 \cdot p_Y(3) \\ &= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2} = 1.5\end{aligned}$$

$$\begin{aligned}\mathbb{V}[Y] &= \sum_{k \in \mathsf{Supp}(Y)} (k - \mu_Y)^2 \cdot p_Y(k) \\ &= (0 - \tfrac{3}{2})^2 \cdot p_Y(0) + (1 - \tfrac{3}{2})^2 \cdot p_Y(1) + (2 - \tfrac{3}{2})^2 \cdot p_Y(2) + (3 - \tfrac{3}{2})^2 \cdot p_Y(3) \\ &= \frac{9}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{3}{8} + \frac{1}{4} \cdot \frac{3}{8} + \frac{9}{4} \cdot \frac{1}{8} = \frac{9}{32} + \frac{3}{32} + \frac{3}{32} + \frac{9}{32} = \frac{24}{32} = \boxed{\frac{3}{4}}\end{aligned}$$

$$\sigma_Y = \sqrt{\mathbb{V}[Y]} = \sqrt{\frac{3}{4}} = \boxed{\frac{\sqrt{3}}{2}} \approx 0.866025$$

An Easier Way to Compute Variance

Computing variances using the definition can be quite tedious!
Fortunately, there's an equivalent formula that's easier to use:

Corollary

(Easier Formula for Variance)

*Let X be a **discrete** random variable with pmf $p_X(k)$. Then:*

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

PROOF: Let $\mu_X \equiv \mathbb{E}[X]$. Then:

$$\mathbb{V}[X] := \sum_{k \in \text{Supp}(X)} (k - \mu_X)^2 \cdot p_X(k) = \sum_{k \in \text{Supp}(X)} (k^2 - 2k\mu_X + \mu_X^2) \cdot p_X(k)$$

$$= \sum_{k \in \text{Supp}(X)} k^2 \cdot p_X(k) - 2\mu_X \cdot \sum_{k \in \text{Supp}(X)} k \cdot p_X(k) + \mu_X^2 \cdot \sum_{k \in \text{Supp}(X)} p_X(k)$$

$$:= \mathbb{E}[X^2] - 2\mu_X \mathbb{E}[X] + \mu_X^2 \cdot 1 = \mathbb{E}[X^2] - 2\mu_X(\mu_X) + \mu_X^2$$

$$= \mathbb{E}[X^2] - 2\mu_X^2 + \mu_X^2 = \mathbb{E}[X^2] - \mu_X^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

QED

Variances of Discrete r.v.'s (Examples)

k	0	1
$p_X(k)$	1/8	7/8

$$\mathbb{E}[X] = \sum_{k \in \text{Supp}(X)} k \cdot p_X(k) = 0 \cdot p_X(0) + 1 \cdot p_X(1) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{7}{8} = \frac{7}{8}$$

$$\mathbb{E}[X^2] = \sum_{k \in \text{Supp}(X)} k^2 \cdot p_X(k) = 0^2 \cdot p_X(0) + 1^2 \cdot p_X(1) = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{7}{8} = \frac{7}{8}$$

$$\therefore \quad \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{7}{8} - \left(\frac{7}{8}\right)^2 = \frac{7}{8} - \frac{49}{64} = \frac{56}{64} - \frac{49}{64} = \boxed{\frac{7}{64}}$$

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k	0	1	2	3
$p_Y(k)$	1/8	3/8	3/8	1/8

$$\mathbb{E}[Y] = \sum_{k \in \text{Supp}(Y)} k \cdot p_Y(k) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2}$$

$$\mathbb{E}[Y^2] = \sum_{k \in \text{Supp}(Y)} k^2 \cdot p_Y(k) = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = 3$$

$$\therefore \quad \mathbb{V}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{12}{4} - \frac{9}{4} = \boxed{\frac{3}{4}}$$

Linearity of Discrete Expected Value (Part I)

When $h(x)$ is a **linear function** [i.e. $h(x) = ax + b$], the following properties make computation of its expected value easier:

Corollary

(Linearity of Expected Value of Discrete Random Variable)

*Let X be a **discrete** random variable with pmf $p_X(k)$. Let $a, b \in \mathbb{R}$. Then:*

$$\mathbb{E}[aX + b] = a \cdot \mathbb{E}[X] + b$$

<u>PROOF:</u>

$$\mathbb{E}[aX + b] \quad := \quad \sum_{k \in \text{Supp}(X)} (ak + b) \cdot p_X(k) = \sum_{k \in \text{Supp}(X)} ak \cdot p_X(k) + \sum_{k \in \text{Supp}(X)} b \cdot p_X(k)$$

$$= \quad a \cdot \left(\sum_{k \in \text{Supp}(X)} k \cdot p_X(k) \right) + b \cdot \left(\sum_{k \in \text{Supp}(X)} p_X(k) \right)$$

$$:= \quad a \cdot \mathbb{E}[X] + b \cdot 1 = a \cdot \mathbb{E}[X] + b$$

QED

Linearity of Discrete Expected Value (Part II)

When $h(x)$ is a **linear function** [i.e. $h(x) = a \cdot g(x) + b$], the following properties make computation of its expected value easier:

Corollary

(Linearity of Expected Value of Discrete Function)

*Let X be a **discrete** random variable with pmf $p_X(k)$. Let $a, b \in \mathbb{R}$. Then:*

$$\mathbb{E}[a \cdot g(X) + b] = a \cdot \mathbb{E}[g(X)] + b$$

PROOF:

$$\mathbb{E}[a \cdot g(X) + b] \quad := \quad \sum_{k \in \text{Supp}(X)} a \cdot g(k) \cdot p_X(k) + \sum_{k \in \text{Supp}(X)} b \cdot p_X(k)$$

$$= \quad a \cdot \left(\sum_{k \in \text{Supp}(X)} g(k) \cdot p_X(k) \right) + b \cdot \left(\sum_{k \in \text{Supp}(X)} p_X(k) \right)$$

$$:= \quad a \cdot \mathbb{E}[g(X)] + b \cdot 1 = a \cdot \mathbb{E}[g(X)] + b$$

QED

Semi-Linearity of Discrete Variance

When $h(x)$ is a **linear function** [i.e. $h(x) = ax + b$], the following properties make computation of its variance easier:

Corollary

(Semi-Linearity of Discrete Variance)

*Let X be a **discrete** random variable with pmf $p_X(k)$. Let $a, b \in \mathbb{R}$. Then:*

$$\mathbb{V}[aX + b] = a^2 \cdot \mathbb{V}[X] \qquad \mathbb{V}[a \cdot g(X) + b] = a^2 \cdot \mathbb{V}[g(X)]$$

PROOF:

$$\begin{aligned}
\mathbb{V}[aX + b] &= \mathbb{E}[(aX + b)^2] - (\mathbb{E}[aX + b])^2 = \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\mathbb{E}[X] + b)^2 \\
&= (a^2\mathbb{E}[X^2] + 2ab\mathbb{E}[X] + b^2) - [a^2(\mathbb{E}[X])^2 + 2ab\mathbb{E}[X] + b^2] \\
&= [a^2\mathbb{E}[X^2] - a^2(\mathbb{E}[X])^2] + (2ab\mathbb{E}[X] - 2ab\mathbb{E}[X]) + (b^2 - b^2) \\
&= a^2[\mathbb{E}[X^2] - (\mathbb{E}[X])^2] + 0 + 0 \\
&= a^2\mathbb{V}[X] \qquad \text{QED}
\end{aligned}$$

Semi-Linearity of Discrete Standard Deviation

When $h(x)$ is a **linear function** [i.e. $h(x) = ax + b$], the following properties make computation of its standard deviation easier:

Corollary

(Semi-Linearity of Discrete Standard Deviation)

*Let X be a **discrete** random variable with pmf $p_X(k)$. Let $a, b \in \mathbb{R}$. Then:*

$$\sigma_{aX+b} = |a| \cdot \sigma_X \qquad \sigma_{a \cdot g(X)+b} = |a| \cdot \sigma_{g(X)}$$

PROOF:

$$\sigma_{aX+b} = \sqrt{\mathbb{V}[aX+b]} = \sqrt{a^2 \mathbb{V}[X]} = \sqrt{a^2} \cdot \sqrt{\mathbb{V}[X]} := |a| \cdot \sigma_X$$

QED

Textbook Logistics for Section 3.3

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Measure of Event	$N(A)$	$\|A\|$
Support of a r.v.	"All possible values of X"	Supp(X)
Support of a r.v.	D	Supp(X)
pmf of a r.v.	$p(x)$	$p_X(k)$
cdf of a r.v.	$F(x)$	$F_X(x)$
Expected Value of a r.v.	$E(X)$	$\mathbb{E}[X]$
Variance of a r.v.	$V(X)$	$\mathbb{V}[X]$

Fin

Fin.