Geometric & Negative Binomial Distributions

Engineering Statistics
Section 3.5

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PART I:

BINOMIAL DISTRIBUTION (REVIEW)

Binomial Random Variables (Applications)

Binomial r. v.'s model the **# successes** of n **independent** Bernoulli Trials:

- Flip n coins, then count # Heads. (Success $\equiv$ "Heads", Failure $\equiv$ "Tails")
- Flip n coins, then count # Tails. (Success $\equiv$ "Tails", Failure $\equiv$ "Heads")

- Roll n dice, then count # 6's. (Success $\equiv$ "6", Failure $\equiv$ "1,2,3,4 or 5")
- Roll n dice, then count # 5's and 6's. (Success $\equiv$ "5 or 6")
- Roll n dice, then count # odd numbers. (Success $\equiv$ "odd #")

- Shake a mixed bag of almonds & cashews until n pieces fall out. Then count how many pieces are almonds. (Success $\equiv$ "almond piece")
- Randomly select n people in a large busy conference. Then count how many are wearing a hat. (Success $\equiv$ person wears hat)
- Randomly select n people in the 'Treatment' group of a medical trial. Then count how many had successful treatment.

- Randomly select n people, count how many will vote 'yes' for resolution.
- Count how many of the next n built widgets are **<u>not</u>** defective.
- Randomly select n websites, count how many got ≥ 1000 views today
- Randomly select n newborn kittens, count how many are female.

<u>NOTE:</u> "Success" & "Failure" are labels – do not interpret them literally.

Binomial Random Variables (Summary)

Proposition

Notation	$X \sim \textit{Binomial}(n,p), \quad n \geq 1, \quad 0 < p < 1, \quad q := 1 - p$
Parameter(s)	$p \equiv \mathbb{P}(\textit{Bernoulli Trial is a Success})$ $q \equiv \mathbb{P}(\textit{Bernoulli Trial is a Failure})$
Support	$\textit{Supp}(X) = \{0, 1, 2, \cdots, n-2, n-1, n\}$
Density (pmf)	$p_X(k; n, p) := \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$
Mean	$\mathbb{E}[X] = np$
Variance	$\mathbb{V}[X] = npq = np(1-p)$
Model(s)	*# Successes of n Bernoulli Trials*
Assumption(s)	*1. Random process comprises of n trials.* *2. Trials are all identical & independent.* *3. Random process has its sample space partitioned into Successes and Failures*

PART II:

GEOMETRIC DISTRIBUTION

Geometric Random Variables (Applications)

Geometric rv's model **# of Bernoulli Trial failures until the 1^{st} success**:

- Flip a coin until a Heads occurs. (Success $\equiv$ "Heads", Failure $\equiv$ "Tails")
- Flip a coin until a Tails occurs. (Success $\equiv$ "Tails", Failure $\equiv$ "Heads")

- Roll a die until a "6" occurs. (Success $\equiv$ "6", Failure $\equiv$ "1,2,3,4 or 5")
- Roll a die until a "5" or a "6" occur. (Success $\equiv$ "5 or 6")
- Roll a die until an odd number occurs. (Success $\equiv$ "odd #")

- Shake a mixed bag of almonds & cashews until one piece falls out. Repeat this until an almond falls out. (Success $\equiv$ "almond piece")
- Randomly select one person in a large busy conference. Repeat this until a person wearing a hat is chosen. (Success $\equiv$ person wears hat)
- Randomly select a person in the 'Treatment' group of a medical trial. Repeat this until a chosen person had a successful treatment.

- Randomly select people until one will vote 'yes' for resolution.
- Build widgets until one is **not** defective.
- Randomly select websites until one got ≥ 1000 views today
- Randomly select newborn kittens until one is female.

NOTE: "Success" & "Failure" are labels – do not interpret them literally.

Geometric Random Variables (Summary)

Proposition

Notation	$X \sim \textit{Geometric}(p), \quad 0 < p < 1, \quad q := 1 - p$
Parameter(s)	$p \quad \equiv \quad$ *Probability of a "Success"*
Support	$\textit{Supp}(X) = \{0, 1, 2, 3, 4, \cdots\}$
pmf	$p_X(k;p) := q^k p = (1-p)^k p$
cdf	$F_X(k;p) = 1 - q^{k+1} = 1 - (1-p)^{k+1}$
Mean	$\mathbb{E}[X] = q/p = (1-p)/p$
Variance	$\mathbb{V}[X] = q/p^2 = (1-p)/p^2$
Model(s)	*# of Bernoulli Trial Failures until the 1^{st} Success occurs*
Assumption(s)	*1. Experiment continues until 1^{st} Success occurs* *2. Trials are all identical & independent.* *3. Random process has its sample space partitioned into Successes and Failures*

PART III:

NEGATIVE BINOMIAL DISTRIBUTION

Negative Binomial Random Variables (Applications)

Negative Binomial rv's model **# of failures until the r^{th} success**:

- Flip a coin until r Heads occur. (Success $\equiv$ "Heads", Failure $\equiv$ "Tails")
- Flip a coin until r Tails occur. (Success $\equiv$ "Tails", Failure $\equiv$ "Heads")

- Roll a die until r 6's occur. (Success $\equiv$ "6", Failure $\equiv$ "1,2,3,4 or 5")
- Roll a die until r 5's or 6's occur. (Success $\equiv$ "5 or 6")
- Roll a die until r odd numbers occur. (Success $\equiv$ "odd #")

- Shake a mixed bag of almonds & cashews until one piece falls out. Repeat this until r almonds fall out. (Success $\equiv$ "almond piece")
- Randomly select one person in a large busy conference. Repeat this until r people wearing a hat are chosen. (Success $\equiv$ person wears hat)
- Randomly select a person in the 'Treatment' group of a medical trial. Repeat this until r chosen people had a successful treatment.

- Randomly select people until r of them will vote 'yes' for resolution.
- Build widgets until r of them are **not** defective.
- Randomly select websites until r of them got ≥ 1000 views today
- Randomly select newborn kittens until r of them are female.

NOTE: "Success" & "Failure" are labels – do not interpret them literally.

Negative Binomial Random Variables (Summary)

Proposition

Notation	$X \sim \textit{NegativeBinomial}(r,p), \quad r > 0, \quad 0 < p < 1, \quad q := 1-p$
Parameter(s)	$r \equiv$ *Number of "Successes"* $p \equiv$ *Probability of a "Success"*
Support	$\textit{Supp}(X) = \{0, 1, 2, 3, 4, \cdots\}$
pmf	$p_X(k; r, p) := \binom{k+r-1}{r-1} p^r q^k = \binom{k+r-1}{r-1} p^r (1-p)^k$
Mean	$\mathbb{E}[X] = rq/p = r(1-p)/p$
Variance	$\mathbb{V}[X] = rq/p^2 = r(1-p)/p^2$
Model(s)	*# of Bernoulli Trial Failures until the r^{th} Success occurs*
Assumption(s)	*1. Experiment continues until r^{th} Success occurs* *2. Trials are all identical & independent.* *3. Random process has its sample space partitioned into Successes and Failures*

NOTE: A Geometric(p) rv is equivalent to a NegativeBinomial$(r = 1, p)$ rv.

Outcomes for Binomial, Geometric, Neg. Binomial rv's

$X \sim$ Binomial(n,p) $\Longrightarrow$ $X \equiv$ (# Successes in n trials)
$X \sim$ Geometric(p) $\Longrightarrow$ $X \equiv$ (# Failures until 1^{st} Success)
$X \sim$ NegativeBinomial(r,p) $\Longrightarrow$ $X \equiv$ (# Failures until r^{th} Success)

VALUE	**OUTCOMES FOR** Binomial$(n=3,p)$	**OUTCOMES FOR** Geometric(p)	**OUTCOMES FOR** NegativeBinomial$(r=2,p)$
$X=0$	FFF	S	SS
$X=1$	SFF, FSF, FFS	FS	FSS, SFS
$X=2$	SSF, SFS, FSS	FFS	$FFSS, FSFS, SFFS$
$X=3$	SSS	$FFFS$	$FFFSS, FFSFS,$ $FSFFS, SFFFS$
$X=4$	(**Not Possible**)	$FFFFS$	$FFFFSS, FFFSFS,$ $FFSFFS, FSFFFS, SFFFFS$
$X=5$	(**Not Possible**)	$FFFFFS$	$FFFFFSS, FFFFSFS,$ $FFFSFFS, FFSFFFS,$ $FSFFFFS, SFFFFFS$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fin.