Exponential & Gamma Distributions

Engineering Statistics
Section 4.4

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PART I:

EXPONENTIAL DISTRIBUTION

Exponential Random Variables (Applications)

Exponential random variables reasonably model **lifetimes** & **inter-arrival waiting times**:

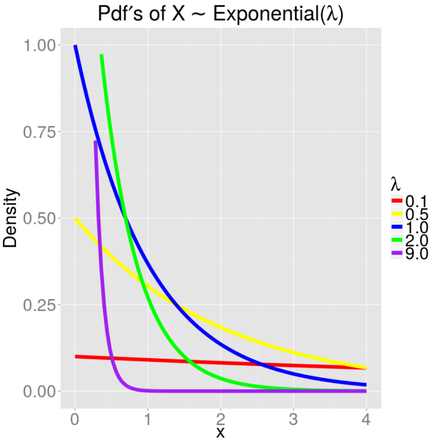
- **Waiting times** between **consecutive arrivals** of a **Poisson process**:
  - Time between radioactive decays of $1\mu g$ of Iodine-123
  - Time between phone calls a dispatcher received
  - Time between emails an account received
  - Time between car accidents at a dangerous intersection
  - Time between insurance claims from a given demographic
  - Time between industrial accidents at a factory
  - Time between wars started in a continent
- **Distances** between **consecutive arrivals** of a **Poisson process**:
  - Distance between mutations of a DNA strand
  - Distance between roadkills on a road near a forest
- **Lifetimes** of electrical components
- Concentrations of air pollutants in a major city
- Blast radii of mining explosives

Exponential Random Variables (Summary)

Proposition

Notation	$X \sim \textit{Exponential}(\lambda), \quad \lambda > 0$
Parameter(s)	$\lambda \equiv$ *Arrival Rate* $(= \alpha$ *of Poisson Process*$)$
Support	$\textit{Supp}(X) = [0, \infty)$
pdf	$f_X(x; \lambda) = \lambda e^{-\lambda x}$
cdf	$F_X(x; \lambda) = \begin{cases} 1 - e^{-\lambda x} & , \text{if } x \geq 0 \\ 0 & , \text{if } x < 0 \end{cases}$
Mean	$\mathbb{E}[X] = 1/\lambda$
Variance	$\mathbb{V}[X] = 1/\lambda^2$
Model(s)	*Waiting times/distances between Poisson arrivals* *Lifetimes of electrical components* *Air pollution concentrations* *Blast radii of mining explosives*

Exponential Density Plots (pdf & cdf)



Verification that Exponential(λ) pdf truly is a valid pdf

Let random variable $X \sim$ Exponential(λ) $\iff f_X(x;\lambda) = f_X(x;\lambda) = \lambda e^{-\lambda x}$

- Non-negativity on its support:

  Observe that $\lambda > 0 \implies \lambda e^{-\lambda x} > 0 \implies f_X(x;\lambda) > 0$

- Universal Integral of Unity:

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)\, dx &= \int_0^{\infty} \lambda e^{-\lambda x}\, dx = \left[-e^{-\lambda x}\right]_{x=0}^{x\to\infty} \\ &\overset{FTC}{=} \left[\lim_{x\to\infty} \left(-e^{-\lambda x}\right)\right] - \left(-e^{-\lambda(0)}\right) \\ &= 0 - (-1) \\ &= 1\end{aligned}$$

Mean of Exponential(λ) random variable (Proof)

Let random variable $X \sim$ Exponential(λ), where $\lambda > 0$ Then:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x;\lambda)\, dx = \int_0^{\infty} \lambda x e^{-\lambda x}\, dx \\
&\overset{IBP}{=} \left[-xe^{-\lambda x} \right]_{x=0}^{x\to\infty} - \int_0^{\infty} \left(-e^{-\lambda x}\right)\, dx \\
&\overset{FTC}{=} [0-0] + \int_0^{\infty} e^{-\lambda x}\, dx \\
&= \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_{x=0}^{x\to\infty} \overset{FTC}{=} 0 - \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda}
\end{aligned}$$

$\therefore \quad \mathbb{E}[X] = \frac{1}{\lambda}$ QED

Variance of Exponential(λ) random variable (Proof)

Let random variable $X \sim$ Exponential(λ), where $\lambda > 0$ Then:

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x;\lambda)\,dx = \int_0^{\infty} \lambda x^2 e^{-\lambda x}\,dx$$

$$\overset{IBP}{=} \Big[-x^2 e^{-\lambda x}\Big]_{x=0}^{x\to\infty} - \int_0^{\infty} \left(-e^{-\lambda x}\right)(2x\,dx) \overset{FTC}{=} [0-0] + \int_0^{\infty} 2xe^{-\lambda x}\,dx$$

$$\overset{IBP}{=} \left[-\frac{2}{\lambda} x e^{-\lambda x}\right]_{x=0}^{x\to\infty} - \int_0^{\infty} \left(-\frac{1}{\lambda} e^{-\lambda x}\right)(2\,dx)$$

$$\overset{FTC}{=} [0-0] + \frac{2}{\lambda}\int_0^{\infty} e^{-\lambda x}\,dx = \frac{2}{\lambda}\left[-\frac{1}{\lambda} e^{-\lambda x}\right]_{x=0}^{x\to\infty}$$

$$\overset{FTC}{=} \frac{2}{\lambda}\left[0 - \left(-\frac{1}{\lambda}\right)\right] = \frac{2}{\lambda^2}$$

$$\therefore \quad \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \qquad \text{QED}$$

Memoryless Property of Exponential Distributions

The Exponential dist. is the only continuous dist. that's **memoryless**: $(s, t > 0)$

$$\begin{aligned}
\mathbb{P}(X > s + t \mid X > s) &= \frac{\mathbb{P}(X > s + t \text{ and } X > s)}{\mathbb{P}(X > s)} \\
&= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{(-\lambda s - \lambda t) + \lambda s} \\
&= e^{-\lambda t} = \mathbb{P}(X > t)
\end{aligned}$$

$$\therefore \quad \mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$$

e.g. $\quad X \sim \text{Exponential}(\lambda) \implies \mathbb{P}(X > 50 \mid X > 20) = \mathbb{P}(X > 30)$

This property means any additional waiting time for an arrival does not depend on or "remember" any amount of waiting time beforehand.

e.g. Waiting 50 min's total for an arrival given 20 min's already elapsed is the same as just waiting 30 min's total for an arrival.

PART II:

GAMMA DISTRIBUTION

Gamma Random Variables (Applications)

Gamma random variables reasonably model:

- Waiting time until α arrivals of Poisson process occur (if α is an **integer**)
- Processes that would typically be modeled by Exponential rv's except...
  - ...the pdf is **unimodal** & **skewed**
  - ...the **memoryless property** is not realized

In fact, an Exponential(λ) rv is equivalently a Gamma$(\alpha = 1, \beta = 1/\lambda)$ rv.

The Gamma Function $\Gamma(\alpha)$ (Definition & Properties)

The **gamma function** is a **special function** that routinely shows up in higher mathematics, combinatorics, physics and (of course) statistics:

Definition

(Gamma Function)

The **gamma function** is defined to be:

$$\Gamma(\alpha) := \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx, \quad \text{where } \alpha > 0$$

Moreover, the gamma function possesses several useful properties:

Corollary

(Useful Properties of the Gamma Function)

- $\Gamma(n) = (n-1)!$ *where n is a **positive integer***
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, *where $\alpha > 0$*

So the gamma function $\Gamma(\alpha)$ is a generalization of the **factorial** $(n!)$.

Gamma Random Variables (Summary)

Proposition

Notation	$X \sim \textit{Gamma}(\alpha, \beta), \quad \alpha, \beta > 0$
Parameter(s)	$\alpha \equiv$ *Shape parameter* $\beta \equiv$ *Scale parameter*
Support	$\textit{Supp}(X) = [0, \infty)$
pdf	$f_X(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$
cdf	$\gamma(x/\beta; \alpha)$
Mean	$\mathbb{E}[X] = \alpha\beta$
Variance	$\mathbb{V}[X] = \alpha\beta^2$
Model(s)	*Time until* α *Poisson arrivals* (*for* ***integer*** α) *Exponential processes* ***w/o memoryless property***
Assumption(s)	(*none*)

A Gamma$(\alpha = 1, \beta = 1/\lambda)$ rv is identical to a Exponential(λ) rv.

Mean of Gamma(α, β) random variable (Proof)

Let $X \sim$ Gamma$(\alpha, \beta) \iff$ pdf $f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$ Then:

$$
\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_0^\infty x \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{(\alpha+1)-1} e^{-x/\beta}\, dx \\
&\overset{CI-1}{=} \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \left(\frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^{\alpha+1}\Gamma(\alpha+1)} \right) x^{(\alpha+1)-1} e^{-x/\beta}\, dx \\
&= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} \int_{\text{Supp}(X)} \underbrace{\frac{1}{\beta^{\alpha+1}\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-x/\beta}}_{\text{pdf of Gamma}(\alpha+1,\beta)}\, dx \\
&= \beta \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \mathbf{1} = \beta \cdot \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta
\end{aligned}
$$

$\therefore \quad \mathbb{E}[X] = \alpha\beta$ QED

Variance of Gamma(α, β) random variable (Proof)

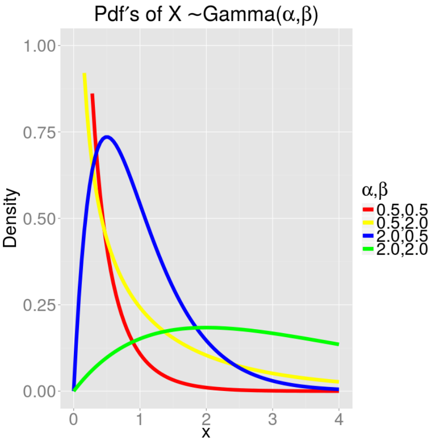
Let $X \sim$ Gamma(α,β) $\iff$ pdf $f_X(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1}e^{-x/\beta}$ Then:

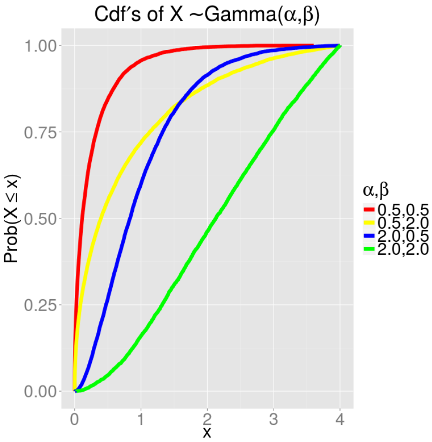
$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx = \int_0^\infty x^2 \cdot \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1}e^{-x/\beta}\,dx \\
&= \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty x^{(\alpha+2)-1}e^{-x/\beta}\,dx \\
&\overset{CI-1}{=} \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty \left(\frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^{\alpha+2}\Gamma(\alpha+2)}\right) x^{(\alpha+2)-1}e^{-x/\beta}\,dx \\
&= \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha\Gamma(\alpha)} \int_{\text{Supp}(X)} \underbrace{\frac{1}{\beta^{\alpha+2}\Gamma(\alpha+2)} x^{(\alpha+2)-1}e^{-x/\beta}}_{\text{pdf of Gamma}(\alpha+2,\beta)}\,dx \\
&= \beta^2 \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \cdot \mathbf{1} = \beta^2 \cdot \frac{\alpha(\alpha+1)\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2 = (\alpha^2+\alpha)\beta^2
\end{aligned}$$

$\therefore\ \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = (\alpha^2+\alpha)\beta^2 - (\alpha\beta)^2 = (\alpha^2+\alpha)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$

QED

Gamma Density Plots (pdf & cdf)





Computing Probabilities involving Gamma r.v.'s

Let $X \sim$ Gamma$(\alpha, \beta) \iff$ pdf $f_X(x) = \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$

Then, $\mathbb{P}(a \le X \le b) = \displaystyle\int_a^b f_X(x)\, dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \underbrace{\int_a^b x^{\alpha-1} e^{-x/\beta}\, dx}_{\text{Nonelementary integral!!}}$

and the cdf is $F_X(x) = \mathbb{P}(X \le x) = \gamma(x/\beta; \alpha) = \dfrac{1}{\Gamma(\alpha)} \underbrace{\displaystyle\int_0^x (t/\beta)^{\alpha-1} e^{-t}\, dt}_{\text{Nonelementary integral!!}}$

The problem with the Gamma distribution is the resulting integrals have no finite closed-form anti-derivative when α is not an integer!!
If α is an integer, tedious use of **integration by parts** is necessary!!

The fix to this is to **numerically** approximate the Gamma **cdf** via a **table**.

Hence, use the table for the **gamma cdf**, which is called the **incomplete Gamma function** and is denoted $\gamma(x; \alpha)$.

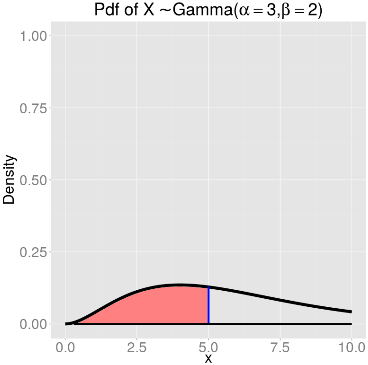
Computing Incomplete Gamma Fcn $\gamma(x;\alpha)$ via Table

x	**Shape Parameter** (α)					
	0.5	**1**	**2**	**3**	**4**	**5**
0.5	0.68269	0.39347	0.09020	0.01439	0.00175	0.00017
1	0.84270	0.63212	0.26424	0.08030	0.01899	0.00366
1.5	0.91674	0.77687	0.44217	0.19115	0.06564	0.01858
2	0.95450	0.86466	0.59399	0.32332	0.14288	0.05265
2.5	0.97465	0.91792	0.71270	0.45619	0.24242	0.10882
3	0.98569	0.95021	0.80085	0.57681	0.35277	0.18474
3.5	0.99185	0.96980	0.86411	0.67915	0.46337	0.27456
4	0.99532	0.98168	0.90842	0.76190	0.56653	0.37116
4.5	0.99730	0.98889	0.93890	0.82642	0.65770	0.46790
5	0.99843	0.99326	0.95957	0.87535	0.73497	0.55951
5.5	0.99909	0.99591	0.97344	0.91162	0.79830	0.64248
6	0.99947	0.99752	0.98265	0.93803	0.84880	0.71494
6.5	0.99969	0.99850	0.98872	0.95696	0.88815	0.77633
7	0.99982	0.99909	0.99270	0.97036	0.91823	0.82701
7.5	0.99989	0.99945	0.99530	0.97974	0.94085	0.86794
8	0.99994	0.99966	0.99698	0.98625	0.95762	0.90037

Computing $\mathbb{P}(X \leq x) = \gamma(x/\beta; \alpha)$ via Table/Software

$$\mathbb{P}(X \leq 5) = \gamma(5/\beta; \alpha) = \gamma(5/2; 3) = \gamma(2.5; 3)$$

x	Shape Parameter (α)			
	0.5	**1**	**2**	**3**
0.5	0.68269	0.39347	0.09020	0.01439
1	0.84270	0.63212	0.26424	0.08030
1.5	0.91674	0.77687	0.44217	0.19115
2	0.95450	0.86466	0.59399	0.32332
2.5	0.97465	0.91792	0.71270	0.45619
3	0.98569	0.95021	0.80085	0.57681
3.5	0.99185	0.96980	0.86411	0.67915
4	0.99532	0.98168	0.90842	0.76190
4.5	0.99730	0.98889	0.93890	0.82642
5	0.99843	0.99326	0.95957	0.87535



TI-8x	(No built-in function)	
TI-36X Pro	(No built-in function)	
MATLAB	`gamcdf(5,3,2)`	(Stats Toolbox)
R	`pgamma(q=5,shape=3,scale=2)`	
Python	`scipy.stats.gamma.cdf(5/2,3)`	(Need SciPy)

Textbook Logistics for Section 4.4

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Support of a r.v.	"All possible values of X"	Supp(X)
pdf of a r.v.	$f(x)$	$f_X(x)$
cdf of a r.v.	$F(x)$	$F_X(x)$
Expected Value of r.v.	$E(X)$	$\mathbb{E}[X]$
Variance of r.v.	$V(X)$	$\mathbb{V}[X]$
Incomplete Gamma Fcn	$F(x/\beta;\alpha)$	$\gamma(x/\beta;\alpha)$

- Skip the "Chi-Squared Distribution" section (pg 174-175)
  - It turns out a Chi-Squared (χ^2) rv is a special kind of Gamma rv.
  - χ^2 distributions will be formally encountered in Chapter 7.
  - This is also used for Statistical Inference of Categorical Data (Chapter 14).

Fin.