Small-Sample CI's for Normal Pop. Variance

Engineering Statistics
Section 7.4

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TTU

06 April 2016

PART I:

HELMERT'S χ^2 DISTRIBUTION

Friedrich Robert Helmert (1843-1917)

Helmert's χ^2 Distribution

Definition

Notation	$X \sim \chi^2_\nu$
Parameter(s)	$\nu \equiv$ # Degrees of Freedom $\quad (\nu = 1, 2, 3, 4, \cdots)$
Support	Supp$(X) = (0, \infty)$
pdf	$f_X(x; \nu) := \frac{1}{2^{\nu/2} \cdot \Gamma(\nu/2)} \cdot x^{(\nu/2)-1} e^{-x/2}$
cdf	$\Phi_{\chi^2}(x; \nu) = \frac{1}{2^{\nu/2} \cdot \Gamma(\nu/2)} \int_0^x \xi^{(\nu/2)-1} e^{-\xi/2} \, d\xi$
Mean	$\mathbb{E}[X] = \nu$
Variance	$\mathbb{V}[X] = 2\nu$
Model(s)	(Used exclusively for Statistical Inference)

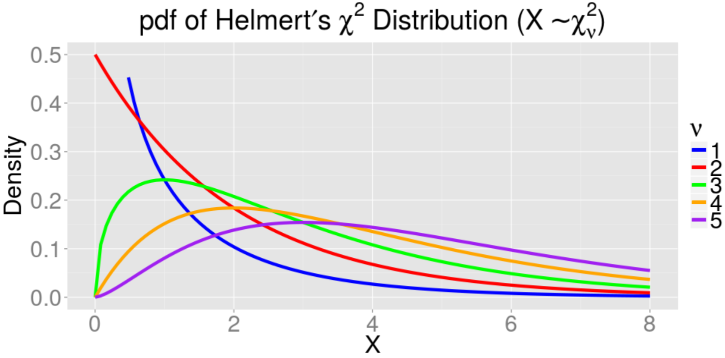
χ is the lowercase Greek letter "chi" (pronounced KEYE)

NOTE: χ^2 distributions are special cases of Gamma distributions.

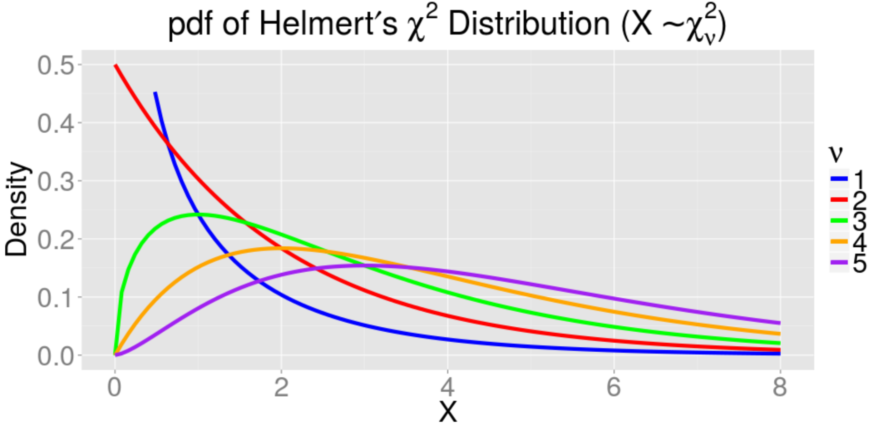
Proposition

Properties of the χ^2_ν distribution:

- *The χ^2_ν pdf curve is positively skewed.*
- *The χ^2_ν pdf curve becomes more symmetric as ν increases.*
- *For $\nu > 40$, the χ^2_ν pdf curve is very close to Normal$(\mu = \nu, \sigma^2 = 2\nu)$ pdf.*



Plots of χ^2 Distributions (A Closer Look)



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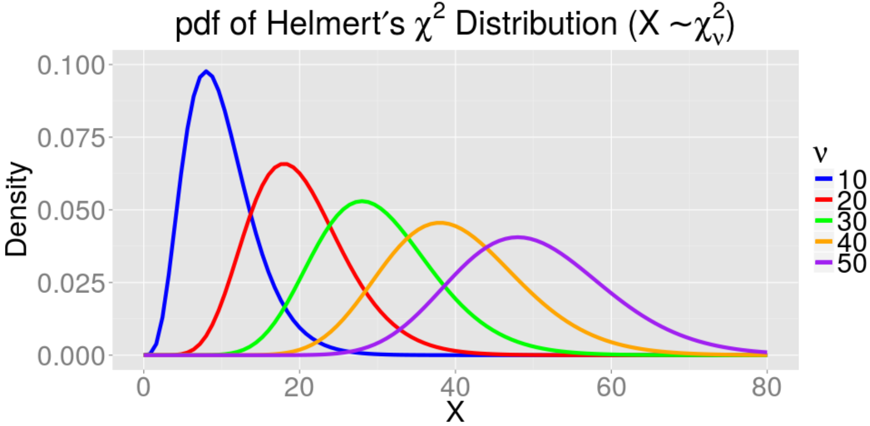


χ^2 Critical Values

Definition

$\chi^{2*}_{\nu,\alpha/2}$ is called a χ^2 **critical value** of the χ^2 distribution with ν df's such that its upper-tail probability is exactly its subscript value $\alpha/2$: (Here, $X \sim \chi^2_\nu$)

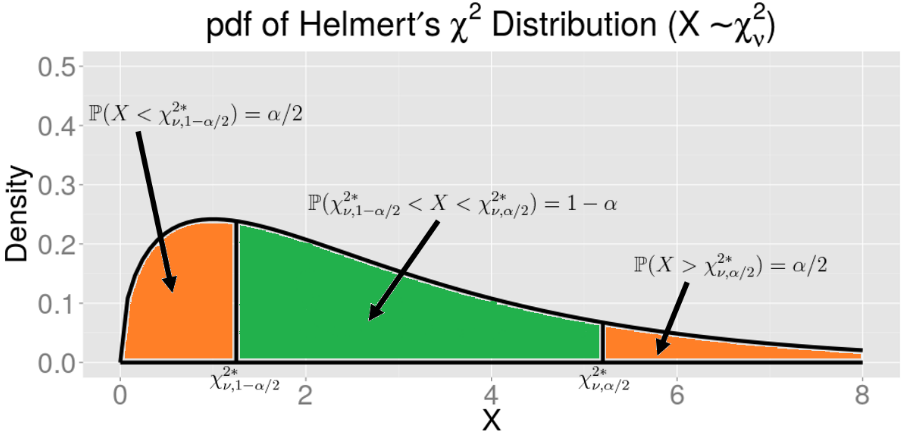
$$\mathbb{P}(X < \chi^{2*}_{\nu,1-\alpha/2}) = \alpha/2 \qquad \mathbb{P}(X > \chi^{2*}_{\nu,\alpha/2}) = \alpha/2$$

IMPORTANT: Do not confuse χ^2 critical values with χ^2 percentiles:

$$\mathbb{P}(X \leq \chi^{2}_{\nu,1-\alpha/2}) = 1 - \alpha/2 \qquad \mathbb{P}(X \leq \chi^{2}_{\nu,\alpha/2}) = \alpha/2$$

Finally, notice that $\chi^{2*}_{\nu,\alpha/2}$ and $\chi^{2*}_{\nu,1-\alpha/2}$ are always **positive**.

χ^2 Critical Values



χ^2 Critical Value Table

	90% CI ($\alpha/2 = 0.050$)		95% CI ($\alpha/2 = 0.025$)		99% CI ($\alpha/2 = 0.005$)	
ν	$\chi^{2*}_{\nu,1-\alpha/2}$	$\chi^{2*}_{\nu,\alpha/2}$	$\chi^{2*}_{\nu,1-\alpha/2}$	$\chi^{2*}_{\nu,\alpha/2}$	$\chi^{2*}_{\nu,1-\alpha/2}$	$\chi^{2*}_{\nu,\alpha/2}$
1	0.004	3.841	0.001	5.024	0.000	7.879
2	0.103	5.991	0.051	7.378	0.010	10.597
3	0.352	7.815	0.216	9.348	0.072	12.838
4	0.711	9.488	0.484	11.143	0.207	14.860
5	1.145	11.070	0.831	12.833	0.412	16.750
6	1.635	12.592	1.237	14.449	0.676	18.548
7	2.167	14.067	1.690	16.013	0.989	20.278
8	2.733	15.507	2.180	17.535	1.344	21.955
9	3.325	16.919	2.700	19.023	1.735	23.589
10	3.940	18.307	3.247	20.483	2.156	25.188
11	4.575	19.675	3.816	21.920	2.603	26.757
12	5.226	21.026	4.404	23.337	3.074	28.300
13	5.892	22.362	5.009	24.736	3.565	29.819
14	6.571	23.685	5.629	26.119	4.075	31.319
15	7.261	24.996	6.262	27.488	4.601	32.801

PART II:

SMALL-SAMPLE CI'S FOR NORMAL POPULATION VARIANCE σ^2
SMALL-SAMPLE CI'S FOR NORMAL POPULATION STD DEV σ

A Statistic related to the χ^2 Distribution

Theorem

Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) *population. Then:*

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

PROOF: (It's complicated...)

Small-Sample CI for Normal Pop. Variance σ^2 (Motivation)

Given a normal population with unknown mean μ and variance σ^2.
Let $\mathbf{X} := (X_1, \ldots, X_n)$ be a random sample from the population.

Then, construct the $100(1-\alpha)\%$ CI for parameter σ^2:

1. Produce a suitable **pivot**: Let $Q(\mathbf{X}; \sigma^2) = \dfrac{(n-1)S^2}{\sigma^2}$
2. Then the pivot is a χ^2 distribution with $\nu = (n-1)$ df's: $Q(\mathbf{X}; \sigma^2) \sim \chi^2_{n-1}$
3. Find constants $a < b$ such that $\mathbb{P}(a < Q(\mathbf{X}; \sigma^2) < b) = 1 - \alpha$

   Since the χ^2_{n-1} pdf is skewed, $a = \chi^{2*}_{n-1,1-\alpha/2}$ and $b = \chi^{2*}_{n-1,\alpha/2}$
4. Manipulate the inequalities to isolate parameter σ^2:
$$\chi^{2*}_{n-1,1-\alpha/2} < \frac{(n-1)S^2}{\sigma^2} < \chi^{2*}_{n-1,\alpha/2} \implies \frac{(n-1)S^2}{\chi^{2*}_{n-1,\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^{2*}_{n-1,1-\alpha/2}}$$
5. Take a size n sample $\mathbf{x} := (x_1, \ldots, x_n)$ from the population.
6. Replace point estimator S with s computed from sample:
$$\frac{(n-1)s^2}{\chi^{2*}_{n-1,\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^{2*}_{n-1,1-\alpha/2}}$$

Small-Sample CI for Normal Pop. Variance & Std Dev

Proposition

Given a normal population with unknown mean μ and variance σ^2.
Let $x_1, \ldots, x_n$ be a small sample taken from the population.

*Then the $100(1-\alpha)\%$ **small-sample CI for** σ^2 is*

$$\left(\frac{(n-1)s^2}{\chi^{2*}_{n-1,\alpha/2}}, \frac{(n-1)s^2}{\chi^{2*}_{n-1,1-\alpha/2}} \right)$$

*Moreover, the $100(1-\alpha)\%$ **small-sample CI for** σ is*

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^{2*}_{n-1,\alpha/2}}}, \sqrt{\frac{(n-1)s^2}{\chi^{2*}_{n-1,1-\alpha/2}}} \right)$$

Textbook Logistics for Section 7.4

- Difference(s) in Notation:

CONCEPT	**TEXTBOOK NOTATION**	**SLIDES/OUTLINE NOTATION**
Probability of Event	$P(A)$	$\mathbb{P}(A)$
z Critical Value	$z_{\alpha/2}$	$z^*_{\alpha/2}$
t Critical Value	$t_{\alpha/2,\nu}$	$t^*_{\nu,\alpha/2}$
χ^2 Critical Value	$\chi^2_{\alpha/2,\nu}$	$\chi^{2*}_{\nu,\alpha/2}$
χ^2 Critical Value	$\chi^2_{1-\alpha/2,\nu}$	$\chi^{2*}_{\nu,1-\alpha/2}$

- Ignore any mention of **one-sided CI's**

Fin

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