Large-Sample z-Tests about Pop. Mean

Engineering Statistics
Section 8.2

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TTU

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PART I:

STANDARD NORMAL CDF TABLE (REVIEW)

Computing Standard Normal cdf $\Phi(z)$ via Table

The **standard normal table** below approximates $\Phi(z) = \mathbb{P}(Z \leq z)$:

z	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53980	0.54380	0.54776	0.55172	0.55567	0.55966	0.56360	0.56749	0.57142	0.57535
0.2	0.57930	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900

Computing $\Phi(z)$ via Table or Calculator or Software

z	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
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1.4	0.91924	0.92073	0.92220	0.92364	[illegible]	0.92647	0.92785	0.92922	0.93056	0.93189
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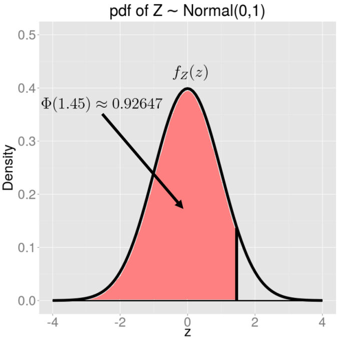

TI-82/83/84+	`normalcdf(-1E99,1.45,0,1)`	2nd → VARS
TI-86	`nmcdf(-1E99,1.45,0,1)`	2nd → MATH
TI-89	Normal cdf	APPS → Stats
TI-36X Pro	Normalcdf	2nd → data
MATLAB	`normcdf(1.45)`	(Stats Toolbox)
R	`pnorm(1.45)`	
Python	`scipy.stats.norm.cdf(1.45)`	(Needs SciPy)

Computing $\Phi(z)$ via Table or Calculator or Software

$$\Phi(-1.23) = 1 - \Phi(1.23) \approx 1 - 0.89065 = 0.10935$$

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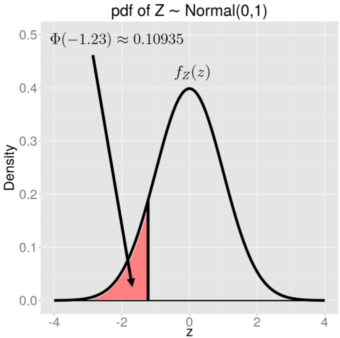

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TI-89	Normal cdf	`APPS` → Stats
TI-36X Pro	Normalcdf	`2nd` → `data`
MATLAB	`normcdf(-1.23)`	(Stats Toolbox)
R	`pnorm(-1.23)`	
Python	`scipy.stats.norm.cdf(-1.23)`	(Needs SciPy)

PART II:

LARGE-SAMPLE HYPOTHESIS TEST
FOR ANY POPULATION MEAN μ

Hypothesis Tests (Procedure)

Proposition

GIVEN: *Random sample* $\mathbf{X} := (X_1, \ldots, X_n)$ *of a population with parameter* θ.
TASK: *Perform a hypothesis test involving parameter* θ.

(1) State the null hypothesis H_0 *and alternative hypothesis* H_A.

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta > \theta_0 \end{aligned} \quad \text{OR} \quad \begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta < \theta_0 \end{aligned} \quad \text{OR} \quad \begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta \neq \theta_0 \end{aligned}$$

(2) Select significance level α: $\alpha = 0.05$ *OR* $\alpha = 0.01$ *OR* $\alpha = 0.001$

(3) Identify the test statistic $W(\mathbf{X}; \theta_0)$

(4) Formulate a decision rule involving P-value & significance level α

If P-value $\leq \alpha$ *then reject* H_0 *in favor of* H_A
If P-value $> \alpha$ *then accept* H_0 (*i.e. fail to reject* H_0)

(5) Obtain a sample $\mathbf{x} := (x_1, \ldots, x_n)$ *from the population*

(6) Compute the test statistic value for the sample $W(\mathbf{x}; \theta_0)$

(7) Compute the corresponding P-value

(8) Use the decision rule to arrive at decision to either reject H_0 *or accept* H_0

Large-Sample z-Test for Pop. Mean (Test Statistic)

Proposition

Given any population with mean μ.
Let $\mathbf{X} := (X_1, X_2, \ldots, X_n)$ be a random sample from the population.
Moreover, let the sample size be "large" meaning $n > 40$.

Suppose an α-level hypothesis test for μ is desired with one of the forms:

$$\begin{aligned} H_0 &: \mu = \mu_0 \\ H_A &: \mu > \mu_0 \end{aligned} \quad \text{OR} \quad \begin{aligned} H_0 &: \mu = \mu_0 \\ H_A &: \mu < \mu_0 \end{aligned} \quad \text{OR} \quad \begin{aligned} H_0 &: \mu = \mu_0 \\ H_A &: \mu \neq \mu_0 \end{aligned}$$

Then, the corresponding test statistic $W(\mathbf{X}; \mu_0)$ is

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \qquad \Longrightarrow \qquad Z \sim \textit{Standard Normal}$$

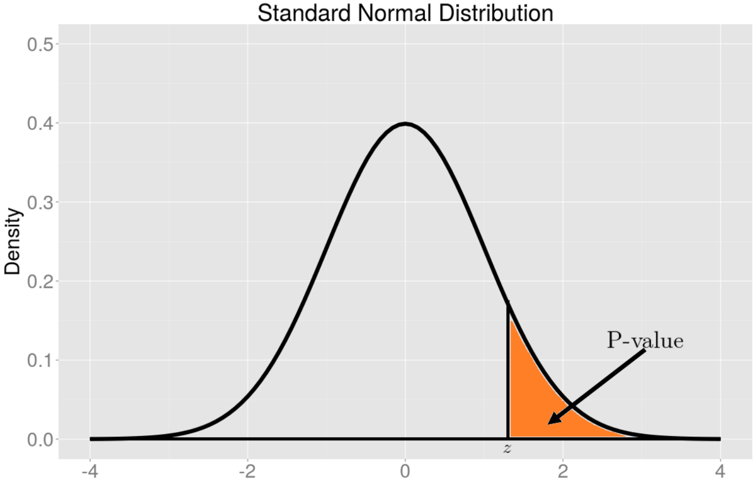
<u>PROOF:</u> Formally, beyond scope of course. ($n > 40$ is needed due to CLT)

Informally, take the corresponding pivot $Q(\mathbf{X}; \mu) := \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$

and replace σ with its point estimator S
and replace μ with its null hypothesis value μ_0

Large-Sample z-Test for Population Mean (P-values)

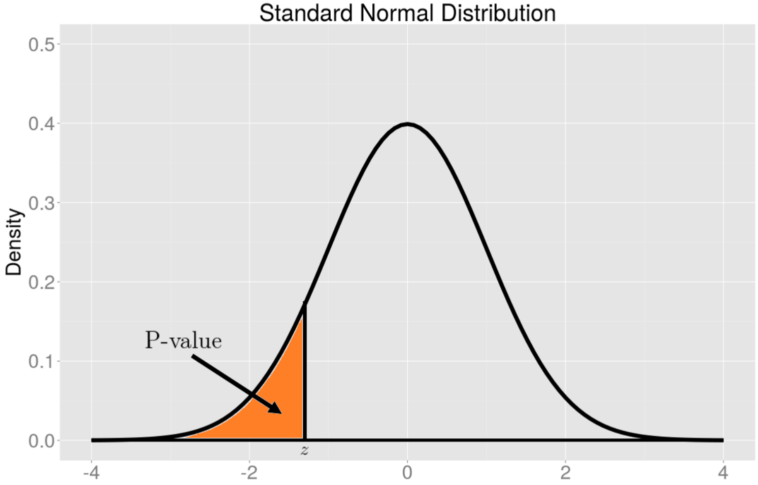
Hypothesis Test: $H_0: \mu = \mu_0$, $H_A: \mu > \mu_0$ (One-Sided z-Test)



P-value $:= \mathbb{P}(\mathbf{W}(X;\mu_0) \geq \mathbf{W}(x;\mu_0)) \approx \mathbb{P}(Z \geq z) = 1 - \mathbb{P}(Z \leq z) = 1 - \Phi(z)$

Large-Sample z-Test for Population Mean (P-values)

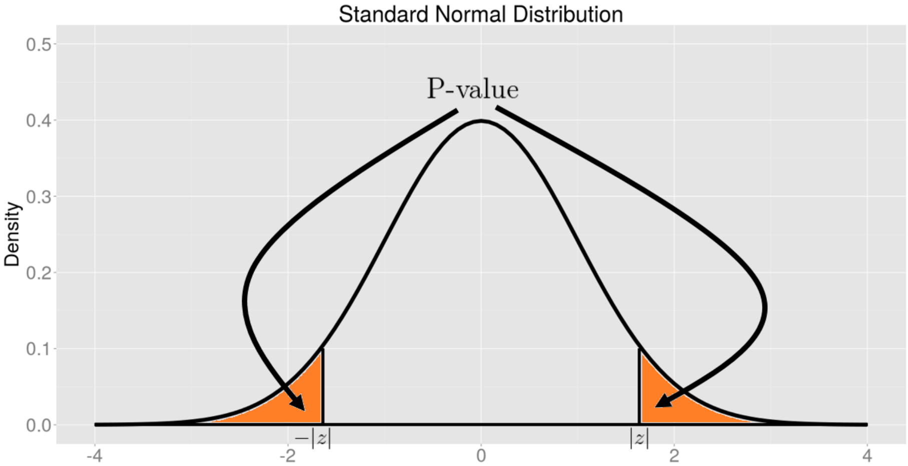
Hypothesis Test: $\begin{array}{l} H_0 : \ \mu = \mu_0 \\ H_A : \ \mu < \mu_0 \end{array}$ (One-Sided z-Test)



$$\text{P-value} := \mathbb{P}(\mathbf{W}(X;\mu_0) \leq \mathbf{W}(x;\mu_0)) \approx \mathbb{P}(Z \leq z) = \Phi(z)$$

Large-Sample z-Test for Population Mean (P-values)

Hypothesis Test: $H_0 : \ \mu = \mu_0$, $H_A : \ \mu \neq \mu_0$ (Two-Sided z-Test)



P-value $\approx \mathbb{P}(|Z| \geq |z|) = \mathbb{P}(Z \leq -|z|) + \mathbb{P}(Z \geq |z|) = 2 \cdot [1 - \Phi(|z|)]$

Large-Sample z-Test for Population Mean (σ unknown)

Proposition

Population:	*Any Population with std dev σ unknown*
Random Sample:	$\mathbf{X} := (X_1, X_2, \ldots, X_n) \quad (n > 40)$
Realized Sample:	$\mathbf{x} := (x_1, x_2, \ldots, x_n) \quad (n > 40)$
Approx. Test Statistic $W(\mathbf{X}; \mu_0)$ *Test Statistic Value* $W(\mathbf{x}; \mu_0)$	$Z = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \qquad z = \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$

HYPOTHESIS TEST:	***P-VALUE DETERMINATION:***
$H_0 : \mu = \mu_0$ vs. $H_A : \mu > \mu_0$	*P-value* $\approx \mathbb{P}(Z \geq z) = 1 - \Phi(z)$
$H_0 : \mu = \mu_0$ vs. $H_A : \mu < \mu_0$	*P-value* $\approx \mathbb{P}(Z \leq z) = \Phi(z)$
$H_0 : \mu = \mu_0$ vs. $H_A : \mu \neq \mu_0$	*P-value* $\approx \mathbb{P}(\lvert Z\rvert \geq \lvert z\rvert) = 2 \cdot [1 - \Phi(\lvert z\rvert)]$

Decision Rule: *If P-value* $\leq \alpha$ *then reject* H_0 *in favor of* H_A
If P-value $> \alpha$ *then accept* H_0 (*i.e. fail to reject* H_0)

Textbook Logistics for Section 8.2

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Alternative Hypothesis	H_a	H_A

- Skip the "Normal Population Dist. with Known σ" section (pg 327-330)
  - In practice, population std deviation σ will not be known a priori.
- Skip the "β and Sample Size Determination" section (pg 330-331)
  - This will be briefly discussed in section 8.5

Fin.