Small-Sample t-Tests about Normal Pop. Mean

Engineering Statistics
Section 8.3

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TTU

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PART I:

GOSSET'S T CDF TABLE

Computing t cdf $\Phi_t(t;\nu)$ via Table

The t **cdf table** below approximates $\Phi_t(t;\nu) = \mathbb{P}(T \le t)$ for $T \sim t_\nu$:

	DEGREES OF FREEDOM (ν)									
t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.1	0.532	0.535	0.537	0.537	0.538	0.538	0.538	0.539	0.539	0.539
0.2	0.563	0.570	0.573	0.574	0.575	0.576	0.576	0.577	0.577	0.577
0.3	0.593	0.604	0.608	0.610	0.612	0.613	0.614	0.614	0.615	0.615
0.4	0.621	0.636	0.642	0.645	0.647	0.648	0.649	0.650	0.651	0.651
0.5	0.648	0.667	0.674	0.678	0.681	0.683	0.684	0.685	0.685	0.686
0.6	0.672	0.695	0.705	0.710	0.713	0.715	0.716	0.717	0.718	0.719
0.7	0.694	0.722	0.733	0.739	0.742	0.745	0.747	0.748	0.749	0.750
0.8	0.715	0.746	0.759	0.766	0.770	0.773	0.775	0.777	0.778	0.779
0.9	0.733	0.768	0.783	0.790	0.795	0.799	0.801	0.803	0.804	0.805
1.0	0.750	0.789	0.804	0.813	0.818	0.822	0.825	0.827	0.828	0.830
1.1	0.765	0.807	0.824	0.833	0.839	0.843	0.846	0.848	0.850	0.851
1.2	0.779	0.823	0.842	0.852	0.858	0.862	0.865	0.868	0.870	0.871
1.3	0.791	0.838	0.858	0.868	0.875	0.879	0.883	0.885	0.887	0.889
1.4	0.803	0.852	0.872	0.883	0.890	0.894	0.898	0.900	0.902	0.904

Computing t cdf $\Phi_t(t;\nu)$ via Table

The t **cdf table** below approximates $\Phi_t(t;\nu) = \mathbb{P}(T \leq t)$ for $T \sim t_\nu$:

	DEGREES OF FREEDOM (ν)									
t	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.1	0.539	0.539	0.539	0.539	0.539	0.539	0.539	0.539	0.539	0.539
0.2	0.578	0.578	0.578	0.578	0.578	0.578	0.578	0.578	0.578	0.578
0.3	0.616	0.616	0.616	0.616	0.616	0.616	0.617	0.617	0.617	0.617
0.4	0.653	0.653	0.653	0.653	0.653	0.653	0.653	0.654	0.654	0.654
0.5	0.688	0.688	0.688	0.689	0.689	0.689	0.689	0.689	0.689	0.689
0.6	0.722	0.722	0.722	0.722	0.722	0.723	0.723	0.723	0.723	0.723
0.7	0.753	0.753	0.754	0.754	0.754	0.754	0.754	0.755	0.755	0.755
0.8	0.782	0.783	0.783	0.783	0.783	0.784	0.784	0.784	0.784	0.784
0.9	0.809	0.810	0.810	0.810	0.811	0.811	0.811	0.811	0.811	0.812
1.0	0.834	0.834	0.835	0.835	0.835	0.836	0.836	0.836	0.836	0.837
1.1	0.856	0.857	0.857	0.857	0.858	0.858	0.858	0.859	0.859	0.859
1.2	0.876	0.877	0.877	0.878	0.878	0.878	0.879	0.879	0.879	0.879
1.3	0.894	0.895	0.895	0.895	0.896	0.896	0.896	0.897	0.897	0.897
1.4	0.910	0.910	0.911	0.911	0.912	0.912	0.912	0.913	0.913	0.913

Computing t cdf $\Phi_t(t;\nu)$ via Table

The t **cdf table** below approximates $\Phi_t(t;\nu) = \mathbb{P}(T \leq t)$ for $T \sim t_\nu$:

	DEGREES OF FREEDOM (ν)									
t	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.1	0.540	0.540	0.540	0.540	0.540	0.540	0.540	0.540	0.540	0.540
0.2	0.579	0.579	0.579	0.579	0.579	0.579	0.579	0.579	0.579	0.579
0.3	0.617	0.617	0.617	0.617	0.617	0.617	0.617	0.617	0.617	0.617
0.4	0.654	0.654	0.654	0.654	0.654	0.654	0.654	0.654	0.654	0.654
0.5	0.690	0.690	0.690	0.690	0.690	0.690	0.690	0.690	0.690	0.690
0.6	0.724	0.724	0.724	0.724	0.724	0.724	0.724	0.724	0.724	0.724
0.7	0.755	0.756	0.756	0.756	0.756	0.756	0.756	0.756	0.756	0.756
0.8	0.785	0.785	0.785	0.785	0.785	0.786	0.786	0.786	0.786	0.786
0.9	0.812	0.813	0.813	0.813	0.813	0.813	0.813	0.813	0.813	0.813
1.0	0.837	0.838	0.838	0.838	0.838	0.838	0.838	0.838	0.838	0.838
1.1	0.860	0.860	0.860	0.860	0.861	0.861	0.861	0.861	0.861	0.861
1.2	0.880	0.881	0.881	0.881	0.881	0.881	0.881	0.881	0.881	0.881
1.3	0.898	0.899	0.899	0.899	0.899	0.899	0.899	0.899	0.899	0.899

PART II:

SMALL-SAMPLE HYPOTHESIS TEST
FOR A NORMAL POPULATION MEAN μ

Hypothesis Tests (Procedure)

Proposition

GIVEN: *Random sample* $\mathbf{X} := (X_1, \ldots, X_n)$ *of a population with parameter* θ.
TASK: *Perform a hypothesis test involving parameter* θ.

(1) State the null hypothesis H_0 *and alternative hypothesis* H_A.

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta > \theta_0 \end{aligned} \quad \textit{OR} \quad \begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta < \theta_0 \end{aligned} \quad \textit{OR} \quad \begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta \neq \theta_0 \end{aligned}$$

(2) Select significance level α*:* $\alpha = 0.05$ *OR* $\alpha = 0.01$ *OR* $\alpha = 0.001$
(3) Identify the test statistic $W(\mathbf{X}; \theta_0)$
(4) Formulate a decision rule involving P-value & significance level α

If P-value $\leq \alpha$ *then reject* H_0 *in favor of* H_A
If P-value $> \alpha$ *then accept* H_0 *(i.e. fail to reject* H_0*)*

(5) Obtain a sample $\mathbf{x} := (x_1, \ldots, x_n)$ *from the population*
(6) Compute the test statistic value for the sample $W(\mathbf{x}; \theta_0)$
(7) Compute the corresponding P-value
(8) Use the decision rule to arrive at decision to either reject H_0 *or accept* H_0

Small-Sample t-Test for Pop. Mean (Test Statistic)

Proposition

Given any population with mean μ.
Let $\mathbf{X} := (X_1, X_2, \ldots, X_n)$ be a random sample from the population.
Suppose an α-level hypothesis test for μ is desired with one of the forms:

$$\begin{array}{ll} H_0: & \mu = \mu_0 \\ H_A: & \mu > \mu_0 \end{array} \quad \text{OR} \quad \begin{array}{ll} H_0: & \mu = \mu_0 \\ H_A: & \mu < \mu_0 \end{array} \quad \text{OR} \quad \begin{array}{ll} H_0: & \mu = \mu_0 \\ H_A: & \mu \neq \mu_0 \end{array}$$

Then, the corresponding test statistic $W(\mathbf{X}; \mu_0)$ is

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \qquad \Longrightarrow \qquad T \sim t_{n-1}$$

PROOF: Formally, beyond scope of course.

Informally, take the corresponding pivot $Q(\mathbf{X}; \mu) := \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$

and replace μ with its null hypothesis value μ_0

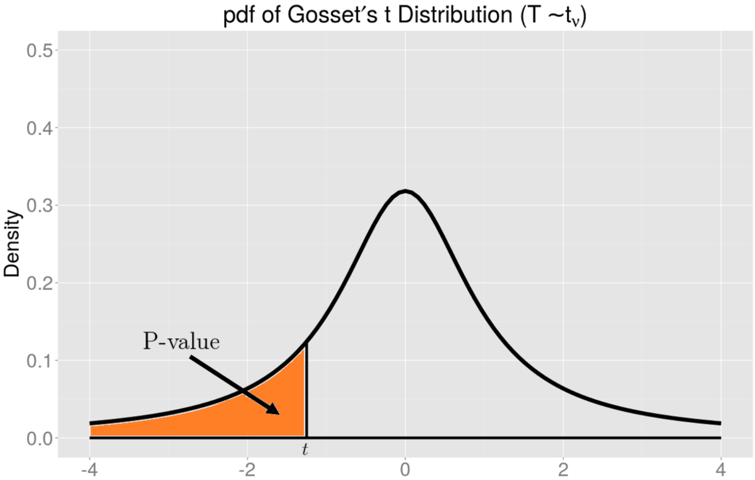
Small-Sample t-Test for Population Mean (P-values)

Hypothesis Test: $H_0: \ \mu = \mu_0$, $H_A: \ \mu > \mu_0$ (One-Sided t-Test)

P-value $:= \mathbb{P}(\mathbf{W}(X;\mu_0) \geq \mathbf{W}(x;\mu_0)) = \mathbb{P}(T \geq t) = 1 - \Phi_t(t;\nu = n-1)$

Small-Sample t-Test for Population Mean (P-values)

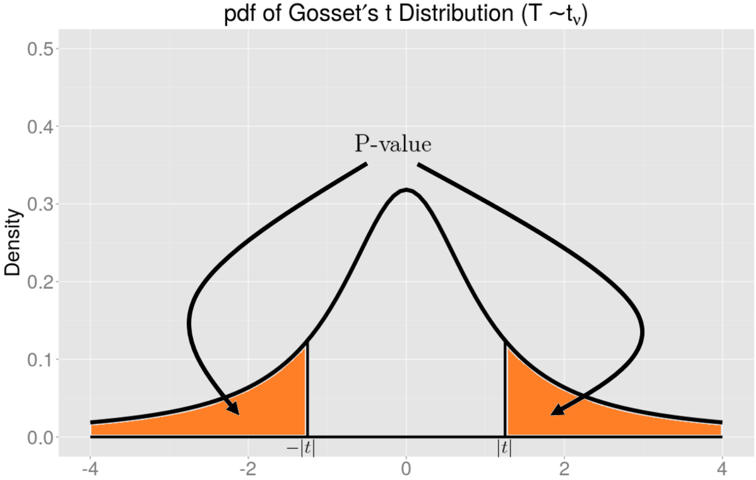
Hypothesis Test: $\begin{array}{l} H_0 : \; \mu = \mu_0 \\ H_A : \; \mu < \mu_0 \end{array}$ (One-Sided t-Test)



P-value $:= \mathbb{P}(\mathbf{W}(X;\mu_0) \leq \mathbf{W}(x;\mu_0)) = \mathbb{P}(T \leq t) = \Phi_t(t; \nu = n-1)$

Small-Sample t-Test for Population Mean (P-values)

Hypothesis Test: $\begin{array}{l} H_0 : \ \mu = \mu_0 \\ H_A : \ \mu \neq \mu_0 \end{array}$ (Two-Sided t-Test)



$$\text{P-value} = \mathbb{P}(|T| \geq |t|) = \mathbb{P}(T \leq -|t|) + \mathbb{P}(T \geq |t|) = 2 \cdot [1 - \Phi_t(|t|; \nu = n-1)]$$

Small-Sample t-Test for Population Mean (σ unknown)

Proposition

Population:	*Normal Population with std dev σ unknown*
Random Sample:	$\mathbf{X} := (X_1, X_2, \ldots, X_n)$
Realized Sample:	$\mathbf{x} := (x_1, x_2, \ldots, x_n)$
Test Statistic $W(\mathbf{X}; \mu_0)$	$T = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}}$
Test Statistic Value $W(\mathbf{x}; \mu_0)$	$t = \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$

HYPOTHESIS TEST:	***P-VALUE DETERMINATION:***
$H_0: \ \mu = \mu_0$ vs. $H_A: \ \mu > \mu_0$	*P-value* $= \mathbb{P}(T \geq t) = 1 - \Phi_t(t; \nu = n-1)$
$H_0: \ \mu = \mu_0$ vs. $H_A: \ \mu < \mu_0$	*P-value* $= \mathbb{P}(T \leq t) = \Phi_t(t; \nu = n-1)$
$H_0: \ \mu = \mu_0$ vs. $H_A: \ \mu \neq \mu_0$	*P-value* $= \mathbb{P}(\lvert T\rvert \geq \lvert t\rvert) = 2 \cdot [1 - \Phi_t(\lvert t\rvert; n-1)]$

Decision Rule:
If P-value $\leq \alpha$ *then reject* H_0 *in favor of* H_A
If P-value $> \alpha$ *then accept* H_0 (*i.e. fail to reject* H_0)

Textbook Logistics for Section 8.3

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Alternative Hypothesis	H_a	H_A

- Skip the "β and Sample Size Determination" section (pg 338-339)
  - This will be briefly discussed in section 8.5
- Skip the **power** of a hypothesis test (pg 340)
  - This will be briefly discussed in section 8.5
- Ignore the "Variation in P-values" section (pg 341-344)
  - This is important, but it's too much for this first Statistics course.

Fin

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