Large-Sample z-Tests about Pop. Proportion

Engineering Statistics
Section 8.4

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15 April 2016

Hypothesis Tests (Procedure)

Proposition

GIVEN: Random sample $\mathbf{X} := (X_1, \ldots, X_n)$ of a population with parameter θ.
TASK: Perform a hypothesis test involving parameter θ.

(1) State the null hypothesis H_0 and alternative hypothesis H_A.

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta > \theta_0 \end{aligned} \quad \text{OR} \quad \begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta < \theta_0 \end{aligned} \quad \text{OR} \quad \begin{aligned} H_0 &: \theta = \theta_0 \\ H_A &: \theta \neq \theta_0 \end{aligned}$$

(2) Select significance level α: $\alpha = 0.05$ OR $\alpha = 0.01$ OR $\alpha = 0.001$
(3) Identify the test statistic $W(\mathbf{X}; \theta_0)$
(4) Formulate a decision rule involving P-value & significance level α

If P-value $\leq \alpha$ then reject H_0 in favor of H_A
If P-value $> \alpha$ then accept H_0 (i.e. fail to reject H_0)

(5) Obtain a sample $\mathbf{x} := (x_1, \ldots, x_n)$ from the population
(6) Compute the test statistic value for the sample $W(\mathbf{x}; \theta_0)$
(7) Compute the corresponding P-value
(8) Use the decision rule to arrive at decision to either reject H_0 or accept H_0

Large-Sample z-Test for Pop. Proportion (Test Stat.)

Proposition

Given any population with proportion p of "successes". $(q := 1 - p)$
Let $\mathbf{X} := (X_1, X_2, \ldots, X_n)$ be a random sample from the population.
Moreover, let sample size n be chosen such that $\min\{np, nq\} \geq 10$

Suppose an α-level hypothesis test for p is desired with one of the forms:

$$\begin{array}{ll} H_0: & p = p_0 \\ H_A: & p > p_0 \end{array} \quad \textit{OR} \quad \begin{array}{ll} H_0: & p = p_0 \\ H_A: & p < p_0 \end{array} \quad \textit{OR} \quad \begin{array}{ll} H_0: & p = p_0 \\ H_A: & p \neq p_0 \end{array}$$

Then, the corresponding test statistic $W(\mathbf{X}; p_0)$ is

$$Z = \frac{(X/n) - p_0}{\sqrt{p_0 q_0 / n}} \quad \textit{where} \quad \begin{array}{ccc} X & \equiv & \#\text{"Successes"} \\ q_0 & := & 1 - p_0 \end{array} \quad \Longrightarrow \quad Z \sim \textit{Std Normal}$$

PROOF: Formally, beyond scope of course.

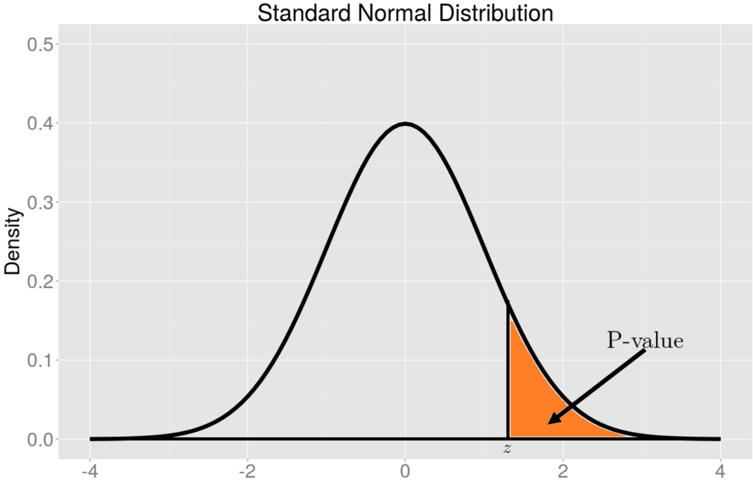
Informally, take the corresponding pivot $Q(\mathbf{X}; p) := \dfrac{(X/n) - p}{\sqrt{pq/n}}$

and replace p with its null hypothesis value p_0 (and q with q_0)

Large-Sample z-Test for Pop. Proportion (P-values)

Hypothesis Test: $H_0 : \; p = p_0$ (One-Sided z-Test)

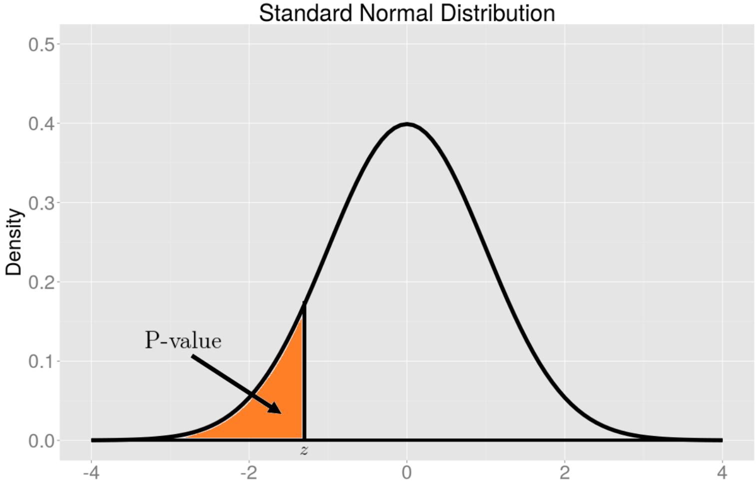
$H_A : \; p > p_0$



P-value $:= \mathbb{P}(\mathbf{W}(X; p_0) \geq \mathbf{W}(x; p_0)) = \mathbb{P}(Z \geq z) = 1 - \Phi(z)$

Large-Sample z-Test for Pop. Proportion (P-values)

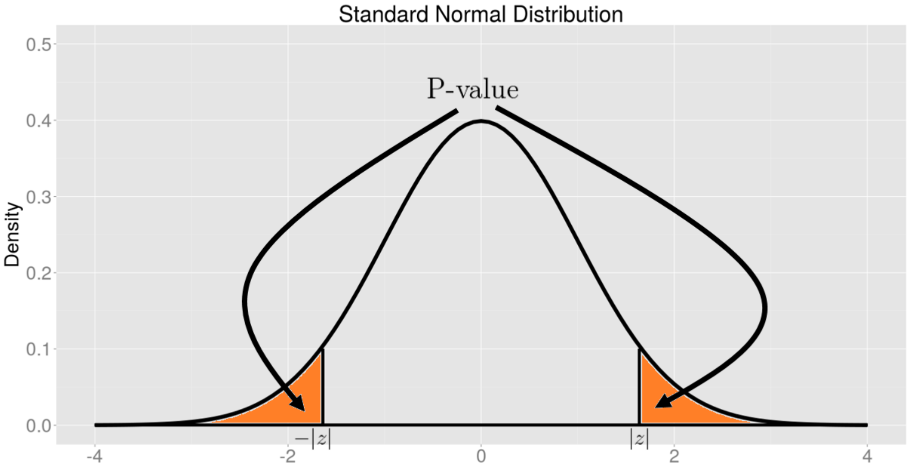
Hypothesis Test: $H_0: \ p = p_0$ $H_A: \ p < p_0$ (One-Sided z-Test)



P-value $:= \mathbb{P}(\mathbf{W}(X;p_0) \leq \mathbf{W}(x;p_0)) = \mathbb{P}(Z \leq z) = \Phi(z)$

Large-Sample z-Test for Pop. Proportion (P-values)

Hypothesis Test: $H_0: p = p_0$, $H_A: p \neq p_0$ (Two-Sided z-Test)



P-value $= \mathbb{P}(|Z| \geq |z|) = \mathbb{P}(Z \leq -|z|) + \mathbb{P}(Z \geq |z|) = 2 \cdot [1 - \Phi(|z|)]$

Large-Sample z-Test for Population Proportion

Proposition

Population:	*Unknown proportion* p *of "successes"*
Random Sample:	$\mathbf{X} := (X_1, X_2, \ldots, X_n)$
Realized Sample:	$\mathbf{x} := (x_1, x_2, \ldots, x_n)$
Test Statistic $W(\mathbf{X}; p_0)$	$Z = \dfrac{(X/n) - p_0}{\sqrt{p_0 q_0 / n}}$
Test Statistic Value $W(\mathbf{x}; p_0)$	$z = \dfrac{\widehat{p} - p_0}{\sqrt{p_0 q_0 / n}}$
	$\widehat{p} := x/n,\ q_0 := 1 - p_0,\ \min\{np_0, nq_0\} \geq 10$
	$x \equiv$ *# "Successes" in realized sample* $\mathbf{x}$

HYPOTHESIS TEST:	***P-VALUE DETERMINATION:***
$H_0 : p = p_0$ *vs.* $H_A : p > p_0$	*P-value* $= \mathbb{P}(Z \geq z) = 1 - \Phi(z)$
$H_0 : p = p_0$ *vs.* $H_A : p < p_0$	*P-value* $= \mathbb{P}(Z \leq z) = \Phi(z)$
$H_0 : p = p_0$ *vs.* $H_A : p \neq p_0$	*P-value* $= \mathbb{P}(\lvert Z\rvert \geq \lvert z\rvert) = 2 \cdot [1 - \Phi(\lvert z\rvert)]$

Decision Rule:
If P-value $\leq \alpha$ *then reject* H_0 *in favor of* H_A
If P-value $> \alpha$ *then accept* H_0 *(i.e. fail to reject* H_0*)*

Textbook Logistics for Section 8.4

- Difference(s) in Notation:

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Probability of Event	$P(A)$	$\mathbb{P}(A)$
Alternative Hypothesis	H_a	H_A

- Skip the "β and Sample Size Determination" section (pg 348-349)
  - This will be briefly discussed in section 8.5

- Skip the "Small-Sample Tests" section (pg 349-350)
  - Instead of the test statistic, X/n, being distributed as a Std Normal, use test statistic, $X \equiv$ (# "Successes"), which is distributed as a Binomial.

Fin

Fin.