EX 10.1.1: Given the following 1-factor balanced experiment:

FACTOR A:	GROUP SIZE:	MEASUREMENTS:
Level 1 ($x_{1\bullet}$)	4	$x_{11},\ x_{12},\ x_{13},\ x_{14}$
Level 2 ($x_{2\bullet}$)	4	$x_{21},\ x_{22},\ x_{23},\ x_{24}$
Level 3 ($x_{3\bullet}$)	4	$x_{31},\ x_{32},\ x_{33},\ x_{34}$

(a) Formulate this experiment as a 1-Factor ANOVA fixed effects linear model.

$$X_{ij} = \mu + \alpha_i^A + E_{ij}, \text{ where } E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2) \qquad \text{for } i = 1,2,3 \text{ and } j = 1,2,3,4$$

(b) Use multivariable calculus to compute the least-squares estimators (LSE's) for this linear model.

Be sure to explain why the constraint $\sum_i \alpha_i = 0$ is necessary to impose.

Find $(\mu; \alpha_1^A, \alpha_2^A, \alpha_3^A) = (\hat{\mu}; \hat{\alpha}_1^A, \hat{\alpha}_2^A, \hat{\alpha}_3^A)$ to minimize $\text{SS}_e := \sum_{i=1}^3 \sum_{j=1}^4 e_{ij}^2 = \sum_{i=1}^3 \sum_{j=1}^4 (x_{ij} - \mu - \alpha_i^A)^2$

Recall from multivariable calculus that the minimum occurs where all first-order partial derivatives are zero:

$$\begin{cases} \frac{\partial}{\partial \mu}[\text{SS}_e]\big|_{(\hat{\mu},\hat{\alpha}_i^A)} = 0 \\ \frac{\partial}{\partial \alpha_i^A}[\text{SS}_e]\big|_{(\hat{\mu},\hat{\alpha}_i^A)} = 0 \end{cases} \implies \begin{cases} -2\cdot\sum_{i=1}^3\sum_{j=1}^4 (x_{ij} - \hat{\mu} - \hat{\alpha}_i^A) &= 0 \\ -2\cdot\sum_{j=1}^4 (x_{ij} - \hat{\mu} - \hat{\alpha}_i^A) &= 0 \end{cases} \text{(Do you understand why } \textstyle\sum_{i=1}^3 \text{ vanishes?)}$$

$$\implies \begin{cases} \sum_{i=1}^3\sum_{j=1}^4 (x_{ij} - \hat{\mu} - \hat{\alpha}_i^A) &= 0 \\ \sum_{j=1}^4 (x_{ij} - \hat{\mu} - \hat{\alpha}_i^A) &= 0 \end{cases} \implies \begin{cases} 12\overline{x}_{\bullet\bullet} - 12\hat{\mu} - 4\sum_{i=1}^3 \hat{\alpha}_i^A &= 0 \\ 4\overline{x}_{i\bullet} - 4\hat{\mu} - 4\hat{\alpha}_i^A &= 0 \end{cases}$$

The fact that $\sum_{i=1}^3 \hat{\alpha}_i^A$ shows up in the linear system requires the general constraint $\sum_i \alpha_i^A = 0$ to be imposed in order to ensure that the linear system has a <u>unique</u> solution for $\hat{\mu}, \hat{\alpha}_1^A, \hat{\alpha}_2^A, \hat{\alpha}_3^A$.

$$\implies \begin{cases} 12\overline{x}_{\bullet\bullet} - 12\hat{\mu} - 4\cdot(0) &= 0 \\ 4\overline{x}_{i\bullet} - 4\hat{\mu} - 4\hat{\alpha}_i^A &= 0 \end{cases} \implies \boxed{\begin{array}{rcl} \hat{\mu} &=& \overline{x}_{\bullet\bullet} \\ \hat{\alpha}_i^A &=& \overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet} \end{array}}$$

(c) Show that each least-squares estimator (LSE) found in part (b) is a linear combination of the data points.

$$\hat{\mu} = \overline{x}_{\bullet\bullet} = \tfrac{1}{12}\sum_{i=1}^3\sum_{j=1}^4 x_{ij}$$

$$\implies \boxed{\hat{\mu} = \tfrac{1}{12}x_{11} + \tfrac{1}{12}x_{12} + \tfrac{1}{12}x_{13} + \tfrac{1}{12}x_{14} + \tfrac{1}{12}x_{21} + \tfrac{1}{12}x_{22} + \tfrac{1}{12}x_{23} + \tfrac{1}{12}x_{24} + \tfrac{1}{12}x_{31} + \tfrac{1}{12}x_{32} + \tfrac{1}{12}x_{33} + \tfrac{1}{12}x_{34}}$$

$$\hat{\alpha}_i^A = \overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet} = \tfrac{1}{4}\sum_{j=1}^4 x_{ij} - \tfrac{1}{12}\sum_{i=1}^3\sum_{j=1}^4 x_{ij}$$

$$\implies \boxed{\hat{\alpha}_1^A = \tfrac{1}{6}x_{11} + \tfrac{1}{6}x_{12} + \tfrac{1}{6}x_{13} + \tfrac{1}{6}x_{14} - \tfrac{1}{12}x_{21} - \tfrac{1}{12}x_{22} - \tfrac{1}{12}x_{23} - \tfrac{1}{12}x_{24} - \tfrac{1}{12}x_{31} - \tfrac{1}{12}x_{32} - \tfrac{1}{12}x_{33} - \tfrac{1}{12}x_{34}}$$

$$\implies \boxed{\hat{\alpha}_2^A = -\tfrac{1}{12}x_{11} - \tfrac{1}{12}x_{12} - \tfrac{1}{12}x_{13} - \tfrac{1}{12}x_{14} + \tfrac{1}{6}x_{21} + \tfrac{1}{6}x_{22} + \tfrac{1}{6}x_{23} + \tfrac{1}{6}x_{24} - \tfrac{1}{12}x_{31} - \tfrac{1}{12}x_{32} - \tfrac{1}{12}x_{33} - \tfrac{1}{12}x_{34}}$$

$$\implies \boxed{\hat{\alpha}_3^A = -\tfrac{1}{12}x_{11} - \tfrac{1}{12}x_{12} - \tfrac{1}{12}x_{13} - \tfrac{1}{12}x_{14} - \tfrac{1}{12}x_{21} - \tfrac{1}{12}x_{22} - \tfrac{1}{12}x_{23} - \tfrac{1}{12}x_{24} + \tfrac{1}{6}x_{31} + \tfrac{1}{6}x_{32} + \tfrac{1}{6}x_{33} + \tfrac{1}{6}x_{34}}$$

(d) Establish the sums of squares partition for the general version of this linear model with I levels each of size J.

$$\begin{aligned}
\text{SS}_A + \text{SS}_{res} &= \textstyle\sum_i\sum_j (\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet})^2 + \sum_i\sum_j (x_{ij} - \overline{x}_{i\bullet})^2 \\
&= \textstyle\sum_i\sum_j (\overline{x}_{i\bullet}^2 - 2\overline{x}_{i\bullet}\overline{x}_{\bullet\bullet} + \overline{x}_{\bullet\bullet}^2) + \sum_i\sum_j (x_{ij}^2 - 2x_{ij}\overline{x}_{i\bullet} + \overline{x}_{i\bullet}^2) \\
&= \textstyle\sum_i\sum_j \overline{x}_{i\bullet}^2 - 2\overline{x}_{\bullet\bullet}\sum_i\sum_j \overline{x}_{i\bullet} + \overline{x}_{\bullet\bullet}^2\sum_i\sum_j 1 + \sum_i\sum_j x_{ij}^2 - 2\sum_i\sum_j x_{ij}\overline{x}_{i\bullet} + \sum_i\sum_j \overline{x}_{i\bullet}^2 \\
&= \textstyle J\sum_i \overline{x}_{i\bullet}^2 - 2J\overline{x}_{\bullet\bullet}\sum_i \overline{x}_{i\bullet} + IJ\overline{x}_{\bullet\bullet}^2 + \sum_i\sum_j x_{ij}^2 - 2\sum_i\sum_j x_{ij}\overline{x}_{i\bullet} + J\sum_i \overline{x}_{i\bullet}^2 \\
&\overset{(1)}{=} \textstyle J\sum_i \overline{x}_{i\bullet}^2 - 2J\overline{x}_{\bullet\bullet}\sum_i \overline{x}_{i\bullet} + IJ\overline{x}_{\bullet\bullet}^2 + \sum_i\sum_j x_{ij}^2 - 2J\sum_i \overline{x}_{i\bullet}^2 + J\sum_i \overline{x}_{i\bullet}^2 \\
&= \textstyle -2J\overline{x}_{\bullet\bullet}\sum_i \overline{x}_{i\bullet} + IJ\overline{x}_{\bullet\bullet}^2 + \sum_i\sum_j x_{ij}^2 \\
&\overset{(2)}{=} \textstyle -2IJ\overline{x}_{\bullet\bullet}^2 + IJ\overline{x}_{\bullet\bullet}^2 + \sum_i\sum_j x_{ij}^2 \\
&= \textstyle \sum_i\sum_j x_{ij}^2 - 2IJ\overline{x}_{\bullet\bullet}^2 + IJ\overline{x}_{\bullet\bullet}^2 \\
&= \textstyle \sum_i\sum_j x_{ij}^2 - 2\sum_i\sum_j x_{ij}\overline{x}_{\bullet\bullet} + \sum_i\sum_j \overline{x}_{\bullet\bullet}^2 \\
&= \textstyle \sum_i\sum_j (x_{ij}^2 - 2x_{ij}\overline{x}_{\bullet\bullet} + \overline{x}_{\bullet\bullet}^2) \\
&= \textstyle \sum_i\sum_j (x_{ij} - \overline{x}_{\bullet\bullet})^2 \\
&= \text{SS}_{total} \qquad\qquad \therefore \boxed{\text{SS}_{total} = \text{SS}_A + \text{SS}_{res}}
\end{aligned}$$

(1) $\overline{x}_{i\bullet} = \frac{1}{J}\sum_j x_{ij}$ (2) $\overline{x}_{\bullet\bullet} = \frac{1}{I}\sum_i \overline{x}_{i\bullet}$

EX 10.1.2: The lifetimes of three light bulb brands were measured and summarized into this table:

BULB BRAND	(BULB LIFETIMES in yrs)
Brand 1 ($x_{1\bullet}$)	9.22, 9.07, 8.95, 8.98, 9.54
Brand 2 ($x_{2\bullet}$)	8.92, 8.88, 9.10, 8.71, 8.85
Brand 3 ($x_{3\bullet}$)	9.08, 8.99, 9.06, 8.93, 9.02

A 1-Factor Balanced Completely Randomized ANOVA (1F bcrANOVA) at significance level $\alpha = 0.01$ is to be performed.

(a) Identify factor A and its levels.

Factor A $\equiv$ **Bulb brand**, Factor A: Level 1 $\equiv$ **Brand 1**, Level 2 $\equiv$ **Brand 2**, Level 3 $\equiv$ **Brand 3**

(b) Determine factor A's level count, I, common group sample size, J, and degrees of freedom, ν_{err} & ν_A.

$I = \boxed{3}$, $J = \boxed{5}$, $\nu_{err} := I(J-1) = 3(5-1) = \boxed{12}$, $\nu_A := I - 1 = 3 - 1 = \boxed{2}$

(c) State the appropriate null hypothesis H_0^A & alternative hypothesis H_A^A.

$H_0^A:\ \mu_1 = \mu_2 = \mu_3$ versus $H_A^A:$ At least two of the μ's differ

(d) Compute the cell means, $\overline{x}_{i\bullet}$.

$$\overline{x}_{1\bullet} := \frac{1}{J}\sum_j x_{1j} = \frac{1}{5}(9.22 + 9.07 + 8.95 + 8.98 + 9.54) = \boxed{9.152}$$
$$\overline{x}_{2\bullet} := \frac{1}{J}\sum_j x_{2j} = \frac{1}{5}(8.92 + 8.88 + 9.10 + 8.71 + 8.85) = \boxed{8.892}$$
$$\overline{x}_{3\bullet} := \frac{1}{J}\sum_j x_{3j} = \frac{1}{5}(9.08 + 8.99 + 9.06 + 8.93 + 9.02) = \boxed{9.016}$$

(e) Compute the cell variances, s_i^2.

$$s_1^2 := \frac{1}{J-1}\sum_j (x_{1j} - \overline{x}_{1\bullet})^2 = \frac{1}{4}\left[(9.22-9.152)^2 + (9.07-9.152)^2 + (8.95-9.152)^2 + (8.98-9.152)^2 + (9.54-9.152)^2\right] = \boxed{0.05807}$$
$$s_2^2 := \frac{1}{J-1}\sum_j (x_{2j} - \overline{x}_{2\bullet})^2 = \frac{1}{4}\left[(8.92-8.892)^2 + (8.88-8.892)^2 + (9.10-8.892)^2 + (8.71-8.892)^2 + (8.85-8.892)^2\right] = \boxed{0.01977}$$
$$s_3^2 := \frac{1}{J-1}\sum_j (x_{3j} - \overline{x}_{3\bullet})^2 = \frac{1}{4}\left[(9.08-9.016)^2 + (8.99-9.016)^2 + (9.06-9.016)^2 + (8.93-9.016)^2 + (9.02-9.016)^2\right] = \boxed{0.00353}$$

(f) Compute the grand mean, $\overline{x}_{\bullet\bullet}$.

$$\overline{x}_{\bullet\bullet} := \frac{1}{I}\sum_i \overline{x}_{i\bullet} = \frac{1}{I}(\overline{x}_{1\bullet} + \overline{x}_{2\bullet} + \overline{x}_{3\bullet}) = \frac{1}{3}(9.152 + 8.892 + 9.016) = \boxed{9.02}$$

(g) Compute the sums of squares, SS_{res} & SS_A.

$$\text{SS}_{res} := \sum_i\sum_j (x_{ij}^{res})^2 = (J-1)\cdot\sum_i s_i^2 = (5-1)\cdot(s_1^2 + s_2^2 + s_3^2) = 4\cdot(0.05807 + 0.01977 + 0.00353) = \boxed{0.32548}$$
$$\text{SS}_A := \sum_i\sum_j (\hat{\alpha}_i^A)^2 = J\cdot\sum_i(\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet})^2 = 5\cdot\left[(9.152-9.02)^2 + (8.892-9.02)^2 + (9.016-9.02)^2\right] = \boxed{0.16912}$$

(h) Compute the square means, MS_{res} & MS_A.

$$\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}} = \frac{0.32548}{12} \approx \boxed{0.02712}, \qquad \text{MS}_A := \frac{\text{SS}_A}{\nu_A} = \frac{0.16912}{2} = \boxed{0.08456}$$

(i) Compute the test F-statistic, f_A, and Fisher's effect size measure, $\hat{\eta}_A^2$.

$$f_A := \frac{\text{MS}_A}{\text{MS}_{res}} = \frac{0.08456}{0.02712} \approx \boxed{3.118}, \qquad \hat{\eta}_A^2 := \frac{\text{SS}_A}{\text{SS}_{total}} = \frac{\text{SS}_A}{\text{SS}_A + \text{SS}_{res}} = \frac{0.16912}{0.16912 + 0.32548} \approx \boxed{0.3419}$$

(j) By hand, lookup F cutoff value, $f^*_{\nu_A,\nu_{res};\alpha}$. By software (SW), compute resulting P-value, p_A.

By hand: $f^*_{\nu_A,\nu_{res};\alpha} = f^*_{2,12;0.01} \overset{LOOKUP}{\approx} \boxed{6.927}$

By SW: $p_A := \mathbb{P}(F > f_A) = 1 - \Phi_F(f_A;\nu_A,\nu_{res}) = 1 - \Phi_F(3.118;\nu_1 = 2, \nu_2 = 12) \overset{SW}{\approx} 1 - 0.9188 = \boxed{0.0812}$

(k) Render the appropriate decision. Also, interpret Fisher's effect size measure, $\hat{\eta}_A^2$.

Since either by hand, $f_A \approx 3.118 < 6.927 \approx f^*_{\nu_A,\nu_{res};\alpha}$, or, by software, $p_A \approx 0.0812 > 0.01 = \alpha$, **accept H_0^A**.
There's not enough evidence from the experiment to claim that at least two of the bulb brands' avg. lifetimes differ.
$\hat{\eta}_A^2 \approx 0.3419 \implies$ About 34% of the variation in the response (bulb lifetime) is due to factor A (bulb brand).

(l) Summarize everything in an 1-Factor ANOVA table.

1-Factor ANOVA Table (Significance Level $\alpha = 0.01$)

Variation Source	df	Sum of Squares	Mean Square	F Stat Value	P-value	Decision
Factor A	2	0.16912	0.08456	3.118	0.0812	Accept H_0^A
Unknown	12	0.32548	0.02712			
Total	14	0.49460				

FYI, Kelley's effect size measure $\hat{\epsilon}_A^2 \approx 0.2322$ and Hays' effect size measure $\hat{\omega}_A^2 \approx 0.2777$

EX 10.1.3: Dentists use resin composites and ceramic fillings among others for cavities in teeth. The shear bond strengths of resin composite-ceramic bonds formed from three possible configurations (conventional, reversed, all-composite) were measured (in MPa) and summarized in the following table:

GROUP:	SAMPLE SIZE:	MEAN:	STD DEV:
Conventional	10	$\overline{x}_{1\bullet} = 10.37$	$s_1 = 1.99$
Reversed	10	$\overline{x}_{2\bullet} = 18.02$	$s_2 = 2.52$
All-Composite	10	$\overline{x}_{3\bullet} = 21.82$	$s_3 = 2.45$

This table and all the details regarding the experiment can be found in the following paper:

A. Della Bona, R. van Noort, "Shear vs. Tensile Bond Strength of Resin Composite Bonded to Ceramic", *Journal of Dental Research*, **74** (1995), 1591-1596.

A 1-Factor Balanced Completely Randomized ANOVA (1F bcrANOVA) at significance level $\alpha = 0.05$ is to be performed.

(a) Identify factor A and its levels.

Factor A $\equiv$ **Resin composite-ceramic bond configurations**

Factor A: Level 1 $\equiv$ **Conventional**, Level 2 $\equiv$ **Reversed**, Level 3 $\equiv$ **All-Composite**

(b) Determine factor A's level count, I, common group sample size, J, and degrees of freedom, ν_{res} & ν_A.

$I = \boxed{3}$, $J = \boxed{10}$, $\nu_{res} := I(J-1) = 3(10-1) = \boxed{27}$, $\nu_A := I - 1 = 3 - 1 = \boxed{2}$

(c) State the appropriate null hypothesis H_0^A & alternative hypothesis H_A^A.

H_0^A : $\mu_1 = \mu_2 = \mu_3$

H_A^A : At least two of the μ's differ

(d) Compute the grand mean, $\overline{x}_{\bullet\bullet}$.

$$\overline{x}_{\bullet\bullet} := \tfrac{1}{I}\sum_i \overline{x}_{i\bullet} = \tfrac{1}{I}\left(\overline{x}_{1\bullet} + \overline{x}_{2\bullet} + \overline{x}_{3\bullet}\right) = \tfrac{1}{3}\left(10.37 + 18.02 + 21.82\right) \approx \boxed{16.737}$$

(e) Compute the sums of squares, SS_{res} & SS_A.

$$\mathrm{SS}_{res} := \sum_i \sum_j (x_{ij}^{res})^2 = (J-1)\cdot\sum_i s_i^2 = 9\cdot\left(1.99^2 + 2.52^2 + 2.45^2\right) = \boxed{146.817}$$

$$\mathrm{SS}_A := \sum_i \sum_j (\hat{\alpha}_i^A)^2 = J\cdot\sum_i \left(\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet}\right)^2 = 10\cdot\left[(10.37 - 16.737)^2 + (18.02 - 16.737)^2 + (21.82 - 16.737)^2\right] = \boxed{680.21667}$$

(f) Compute the square means, MS_{res} & MS_A.

$$\mathrm{MS}_{res} := \frac{\mathrm{SS}_{res}}{\nu_{res}} = \frac{146.817}{27} \approx \boxed{5.4377}, \qquad \mathrm{MS}_A := \frac{\mathrm{SS}_A}{\nu_A} = \frac{680.21667}{2} = \boxed{340.108335}$$

(g) Compute the test F-statistic, f_A, and Fisher's effect size measure, $\hat{\eta}_A^2$.

$$f_A := \frac{\mathrm{MS}_A}{\mathrm{MS}_{res}} = \frac{340.108335}{5.4377} \approx \boxed{62.546}, \qquad \hat{\eta}_A^2 := \frac{\mathrm{SS}_A}{\mathrm{SS}_{total}} = \frac{\mathrm{SS}_A}{\mathrm{SS}_A + \mathrm{SS}_{res}} = \frac{680.21667}{680.21667 + 146.817} \approx \boxed{0.8225}$$

(h) By hand, lookup F cutoff value, $f^*_{\nu_A,\nu_{res};\alpha}$. By software (SW), compute resulting P-value, p_A.

By hand: $f^*_{\nu_A,\nu_{res};\alpha} = f^*_{2,27;0.05} \overset{LOOKUP}{\approx} \boxed{3.354}$

By SW: $p_A := \mathbb{P}(F > f_A) = 1 - \Phi_F(f_A; \nu_A, \nu_{res}) = 1 - \Phi_F(62.546; \nu_1 = 2, \nu_2 = 27) \overset{SW}{\approx} 1 - 0.999999999927 = \boxed{7.3 \times 10^{-11}}$

(i) Render the appropriate decision. Also, interpret Fisher's effect size measure, $\hat{\eta}_A^2$.

Since either by hand, $f_A \approx 62.546 > 3.354 \approx f^*_{\nu_A,\nu_{res};\alpha}$, or, by software, $p_A \approx 7.3 \times 10^{-11} < 0.05 = \alpha$, **reject H_0^A**.
There's enough evidence from the experiment to claim that at least two of the bond configs' avg. shear bond strengths differ.
$\hat{\eta}_A^2 \approx 0.8225 \implies$ About 82% of the variation in the response (shear bond strength) is due to factor A (bond config).

(j) Summarize everything in an 1-Factor ANOVA table.

1-Factor ANOVA Table (Significance Level $\alpha = 0.05$)

Variation Source	df	Sum of Squares	Mean Square	F Stat Value	P-value	Decision
Factor A	2	680.217	340.11	62.546	7.3×10^{-11}	Reject H_0^A
Unknown	27	146.817	5.44			
Total	29	827.034				

FYI, Kelley's effect size measure $\hat{\epsilon}_A^2 \approx 0.8093$ and Hays' effect size measure $\hat{\omega}_A^2 \approx 0.8040$