EX 10.3.1: The lifetimes of three light bulb brands were measured:

BULB BRAND	(BULB LIFETIMES in yrs)
Brand 1 ($x_{1\bullet}$)	9.22, 9.07, 8.95, 8.98, 9.54
Brand 2 ($x_{2\bullet}$)	8.92, 8.88, 9.10
Brand 3 ($x_{3\bullet}$)	9.08, 8.99, 9.06, 8.93

(a) Formulate this experiment as a 1-Factor ANOVA fixed effects linear model.

$X_{ij} = \mu + \alpha_i^A + E_{ij}$ where

$\mu \equiv$ Common population mean lifetime of the three bulb brands

$\alpha_i^A \equiv$ Lifetime deviation from μ due to Brand i bulb

$E_{ij} \equiv$ Lifetime deviation from μ due to random error/noise

(b) Perform the appropriate 1-Factor ANOVA at significance level $\alpha = 0.01$ – compute both the F-cutoff and P-value.

$I = 3;\ J_1 = 5,\ J_2 = 3,\ J_3 = 4 \implies n = \sum_i J_i = 5+3+4 = 12 \implies \nu_{res} = n - I = 12-3 = 9,\ \ \nu_A = I - 1 = 3-1 = 2$

$$\begin{aligned}
\overline{x}_{1\bullet} &:= \tfrac{1}{J_1}\textstyle\sum_{j=1}^{J_1} x_{1j} = \tfrac{1}{5}(9.22+9.07+8.95+8.98+9.54) = 9.152\\
\overline{x}_{2\bullet} &:= \tfrac{1}{J_2}\textstyle\sum_{j=1}^{J_2} x_{2j} = \tfrac{1}{3}(8.92+8.88+9.10) \approx 8.967\\
\overline{x}_{3\bullet} &:= \tfrac{1}{J_3}\textstyle\sum_{j=1}^{J_3} x_{3j} = \tfrac{1}{4}(9.08+8.99+9.06+8.93) = 9.015\\
\overline{x}_{\bullet\bullet} &:= \tfrac{1}{I}\textstyle\sum_i \overline{x}_{i\bullet} = \tfrac{1}{3}(9.152+8.967+9.015) \approx 9.045
\end{aligned}$$

$$s_1^2 := \tfrac{1}{J_1-1}\textstyle\sum_{j=1}^{J_1}(x_{1j}-\overline{x}_{1\bullet})^2 = \dfrac{1}{5-1}\left[\begin{array}{l}(9.22-9.152)^2+(9.07-9.152)^2+(8.95-9.152)^2\\ +(8.98-9.152)^2+(9.54-9.152)^2\end{array}\right] = 0.05807$$

$$s_2^2 := \tfrac{1}{J_2-1}\textstyle\sum_{j=1}^{J_2}(x_{2j}-\overline{x}_{2\bullet})^2 = \dfrac{1}{3-1}\left[(8.92-8.967)^2+(8.88-8.967)^2+(9.10-8.967)^2\right] = 0.01373$$

$$s_3^2 := \tfrac{1}{J_3-1}\textstyle\sum_{j=1}^{J_3}(x_{3j}-\overline{x}_{3\bullet})^2 = \dfrac{1}{4-1}\left[\begin{array}{l}(9.08-9.015)^2+(8.99-9.015)^2+(9.06-9.015)^2\\ +(8.93-9.015)^2\end{array}\right] \approx 0.0047$$

$$\begin{aligned}
\text{SS}_{res} &:= \textstyle\sum_i (J_i-1)s_i^2 = (5-1)(0.05807)+(3-1)(0.01373)+(4-1)(0.0047) = 0.27384\\
\text{SS}_A &:= \textstyle\sum_i J_i(\overline{x}_{i\bullet}-\overline{x}_{\bullet\bullet})^2 = 5(9.152-9.045)^2+3(8.967-9.045)^2+4(9.015-9.045)^2 = 0.0791
\end{aligned}$$

$$\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}} = \frac{0.27384}{9} = 0.0304,\quad \text{MS}_A := \frac{\text{SS}_A}{\nu_A} = \frac{0.0791}{2} = 0.03955$$

$$f_A := \frac{\text{MS}_A}{\text{MS}_{res}} = \frac{0.03955}{0.0304} \approx \mathbf{1.301}$$

By hand: $f^*_{\nu_A,\nu_{res};\alpha} = f^*_{2,9;0.01} \overset{LOOKUP}{\approx} \mathbf{8.02}$

By SW: $p_A := \mathbb{P}(F > f_A) = 1 - \Phi_F(f_A;\nu_A,\nu_{res}) = 1 - \Phi_F(1.301;\nu_1=2,\nu_2=9) \overset{SW}{\approx} 1 - 0.6811 \approx \mathbf{0.3189}$

Since either by hand, $f_A \approx 1.301 < 8.02 \approx f^*_{\nu_A,\nu_{res};\alpha}$, or, by software, $p_A \approx 0.3189 > 0.01 = \alpha$, **accept H_0^A**.
There's not enough evidence from the experiment to claim that at least two of the bulb brands' avg. lifetimes differ.

(c) Summarize everything in an 1-Factor ANOVA table.

1-Factor ANOVA Table (Significance Level $\alpha = 0.01$)						
Variation Source	**df**	**Sum of Squares**	**Mean Square**	**F Stat Value**	**P-value**	**Decision**
Factor A	2	0.07910	0.03955	1.301	0.3189	Accept H_0^A
Unknown	9	0.27384	0.0304			
Total	11	0.35294				

EX 10.3.2: Dentists use resin composites and ceramic fillings among others for cavities in teeth. The shear bond strengths of resin composite-ceramic bonds formed from three possible configurations (conventional, all-composite, reversed) were measured (in MPa) and summarized in the following table:

GROUP:	SAMPLE SIZE:	MEAN:	STD DEV:
Conventional	9	$\overline{x}_{1\bullet} = 10.37$	$s_1 = 1.99$
All-Composite	8	$\overline{x}_{2\bullet} = 21.82$	$s_2 = 2.45$
Reversed	6	$\overline{x}_{3\bullet} = 18.02$	$s_3 = 2.52$

This table and all the details regarding the experiment can be found in the following paper:

A. Della Bona, R. van Noort, "Shear vs. Tensile Bond Strength of Resin Composite Bonded to Ceramic", *Journal of Dental Research*, **74** (1995), 1591-1596.

(a) Formulate this experiment as a 1-Factor ANOVA fixed effects linear model.

$$X_{ij} = \mu + \alpha_i^A + E_{ij} \quad \text{where} \quad \begin{aligned} \mu &\equiv \text{Common population mean shear bond strength of the three configurations} \\ \alpha_i^A &\equiv \text{Shear bond strength deviation from } \mu \text{ due to } i^{th} \text{ configuration} \\ E_{ij} &\equiv \text{Shear bond strength deviation from } \mu \text{ due to random error/noise} \end{aligned}$$

(b) Perform the appropriate 1-Factor ANOVA at significance level $\alpha = 0.05$ – compute both the F-cutoff and P-value.

$I = 3;\ J_1 = 9,\ J_2 = 6,\ J_3 = 8 \implies n = \sum_i J_i = 9+6+8 = 23 \implies \nu_{res} = n - I = 23-3 = 20,\ \ \nu_A = I-1 = 3-1 = 2$

$$\begin{aligned} \overline{x}_{\bullet\bullet} &:= \tfrac{1}{I}\sum_i \overline{x}_{i\bullet} = \tfrac{1}{3}(10.37 + 21.82 + 18.02) \approx 16.737 \\ \text{SS}_{res} &:= \sum_i (J_i - 1)s_i^2 = (9-1)\cdot 1.99^2 + (8-1)\cdot 2.45^2 + (6-1)\cdot 2.52^2 = 105.4503 \\ \text{SS}_A &:= \sum_i J_i(\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet})^2 = 9\cdot(10.37-16.737)^2 + 8\cdot(21.82-16.737)^2 + 6\cdot(18.02-16.737)^2 = 581.4198 \end{aligned}$$

$$\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}} = \frac{105.4503}{20} = 5.272515, \quad \text{MS}_A := \frac{\text{SS}_A}{\nu_A} = \frac{581.4198}{2} = 290.7099$$

$$f_A := \frac{\text{MS}_A}{\text{MS}_{res}} = \frac{290.7099}{5.272515} \approx \mathbf{55.1369}$$

By hand: $f^*_{\nu_A,\nu_{res};\alpha} = f^*_{2,20;0.05} \overset{LOOKUP}{\approx} \mathbf{3.49}$

By SW: $p_A := \mathbb{P}(F > f_A) = 1 - \Phi_F(f_A;\nu_A,\nu_{res}) = 1 - \Phi_F(55.1369;\nu_1 = 2, \nu_2 = 20) \overset{SW}{\approx} 1 - 0.999999992727 \approx \mathbf{7.273 \times 10^{-9}}$

Since either by hand, $f_A \approx 55.1369 \gg 3.49 \approx f^*_{\nu_A,\nu_{res};\alpha}$, or, by software, $p_A \approx 7.273\times 10^{-9} \ll 0.05 = \alpha$, $\boxed{\textbf{reject } H_0^A}$.
There's enough evidence from the experiment to claim that at least two of the bond configs' avg. shear bond strengths differ.

(c) Perform the appropriate Tukey-Kramer Complete Pairwise Post-Hoc Comparison.

$$\begin{aligned} w_{(12)} &= q^*_{I,\nu_{res};\alpha}\cdot\sqrt{\text{MS}_{res}\cdot\tfrac{1}{2}\left(\tfrac{1}{J_1}+\tfrac{1}{J_3}\right)} \approx q^*_{3,20;0.05}\cdot\sqrt{5.272515\cdot\tfrac{1}{2}\left(\tfrac{1}{9}+\tfrac{1}{8}\right)} \overset{LOOKUP}{\approx} \mathbf{3.58}\cdot\sqrt{5.272515\cdot\tfrac{17}{144}} \approx \mathbf{2.824} \\ w_{(13)} &= q^*_{I,\nu_{res};\alpha}\cdot\sqrt{\text{MS}_{res}\cdot\tfrac{1}{2}\left(\tfrac{1}{J_1}+\tfrac{1}{J_2}\right)} \approx q^*_{3,20;0.05}\cdot\sqrt{5.272515\cdot\tfrac{1}{2}\left(\tfrac{1}{9}+\tfrac{1}{6}\right)} \overset{LOOKUP}{\approx} \mathbf{3.58}\cdot\sqrt{5.272515\cdot\tfrac{5}{36}} \approx \mathbf{3.064} \\ w_{(23)} &= q^*_{I,\nu_{res};\alpha}\cdot\sqrt{\text{MS}_{res}\cdot\tfrac{1}{2}\left(\tfrac{1}{J_2}+\tfrac{1}{J_3}\right)} \approx q^*_{3,20;0.05}\cdot\sqrt{5.272515\cdot\tfrac{1}{2}\left(\tfrac{1}{8}+\tfrac{1}{6}\right)} \overset{LOOKUP}{\approx} \mathbf{3.58}\cdot\sqrt{5.272515\cdot\tfrac{7}{48}} \approx \mathbf{3.139} \end{aligned}$$

$$\overline{x}_{(2)\bullet} \notin \left(\overline{x}_{(1)\bullet},\ \overline{x}_{(1)\bullet} + w_{(12)}\right), \qquad \overline{x}_{(3)\bullet} \notin \left(\overline{x}_{(1)\bullet},\ \overline{x}_{(1)\bullet} + w_{(13)}\right), \qquad \overline{x}_{(3)\bullet} \in \left(\overline{x}_{(2)\bullet},\ \overline{x}_{(2)\bullet} + w_{(23)}\right)$$

$\overline{x}_{(1)\bullet}$	$\overline{x}_{(2)\bullet}$	$\overline{x}_{(3)\bullet}$
$\overline{x}_{1\bullet}$	$\overline{x}_{3\bullet}$	$\overline{x}_{2\bullet}$
10.37	<u>18.02</u>	<u>21.82</u>

The experiment suggests that the all-composite and reversed configurations each have a significantly higher shear bond strength than the conventional.

The experiment suggests that there is not a significant difference in shear bond strength between the all-composite and reversed configurations.