1-FACTOR FIXED EFFECTS UNBALANCED LINEAR MODELS [DEVORE 10.3]

1-FACTOR FIXED EFFECTS UNBALANCED LINEAR (STATISTICAL) MODEL (DEFINITION):

Given a 1-factor unbalanced experiment with $I > 2$ groups, each of size J_i.

Let $X_{ij} \equiv$ random variable for j^{th} measurement in the i^{th} group.

Then, the **fixed effects linear (statistical) model** for the experiment is defined as:

$$X_{ij} = \mu + \alpha_i^A + E_{ij} \qquad \text{where} \qquad E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$$

$\mu \equiv$ population grand mean of all I population means

$\alpha_i^A \equiv$ deviation of i^{th} population mean μ_i from μ due to Factor A

$E_{ij} \equiv$ rv for error/noise applied to j^{th} measurement in i^{th} group

Fixed effects means all relevant levels of factor A are considered in model.

1-FACTOR LINEAR MODEL (MOTIVATING EXAMPLES):

$$X_{ij} = \mu$$

$$\mu := 3.2$$

$$\mu_1 = 3.2,\ \mu_2 = 3.2,\ \mu_3 = 3.2$$

FACTOR A:	MEASUREMENTS:			
Level 1 ($x_{1\bullet}$)	$x_{11} = 3.2$,	$x_{12} = 3.2$,	$x_{13} = 3.2$	
Level 2 ($x_{2\bullet}$)	$x_{21} = 3.2$,	$x_{22} = 3.2$,	$x_{23} = 3.2$,	$x_{24} = 3.2$
Level 3 ($x_{3\bullet}$)	$x_{31} = 3.2$,	$x_{32} = 3.2$		

$$X_{ij} = \mu + \alpha_i^A$$

$$\mu := 3.2$$

$$\alpha_1^A := -5.5,\ \alpha_2^A := -2.0,\ \alpha_3^A := 7.5$$

$$\mu_1 = -2.3,\ \mu_2 = 1.2,\ \mu_3 = 10.7$$

FACTOR A:	MEASUREMENTS:			
Level 1 ($x_{1\bullet}$)	$x_{11} = -2.3$,	$x_{12} = -2.3$,	$x_{13} = -2.3$	
Level 2 ($x_{2\bullet}$)	$x_{21} = 1.2$,	$x_{22} = 1.2$,	$x_{23} = 1.2$,	$x_{24} = 1.2$
Level 3 ($x_{3\bullet}$)	$x_{31} = 10.7$,	$x_{32} = 10.7$		

$$X_{ij} = \mu + \alpha_i^A + E_{ij}$$

$$\mu := 3.2,\ \alpha_1^A := -5.5,\ \alpha_2^A := -2.0,\ \alpha_3^A := 7.5,\ E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2 := 3.24)$$

$$\mu_1 = -2.3,\ \mu_2 = 1.2,\ \mu_3 = 10.7$$

FACTOR A:	MEASUREMENTS:			
Level 1 ($x_{1\bullet}$)	$x_{11} = -1.23$,	$x_{12} = -1.17$,	$x_{13} = 0.05$	
Level 2 ($x_{2\bullet}$)	$x_{21} = 0.54$,	$x_{22} = 1.03$,	$x_{23} = 0.62$,	$x_{24} = 1.63$
Level 3 ($x_{3\bullet}$)	$x_{31} = 13.64$,	$x_{32} = 12.30$		

1F UNBALANCED LINEAR MODELS (POINT ESTIMATORS) [DEVORE 10.3]

1-FACTOR UNBALANCED LINEAR MODEL (LEAST-SQUARES ESTIMATORS – LSE's):

Given a 1-factor unbalanced linear model: $X_{ij} = \mu + \alpha_i^A + E_{ij}$ where $E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$ Then:

(a) The **least-squares♠♣ estimators†‡ (LSE's)** for the model parameters are:

$$\begin{aligned} \hat{\mu} &= \overline{x}_{\bullet\bullet} \\ \hat{\alpha}_i^A &= \overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet} \end{aligned} \quad \text{where} \quad \begin{aligned} \overline{x}_{\bullet\bullet} &\equiv \text{Grand sample mean} \\ \overline{x}_{i\bullet} &\equiv \text{Sample mean of } i^{th} \text{ group} \end{aligned}$$

(b) For these least-squares estimators, it's required that $\sum_i J_i \hat{\alpha}_i^A = 0$.

(c) These least-squares estimators are all unbiased.

†A. Dean, D. Voss, D. Draguljić, *Design & Analysis of Experiments*, 2^{nd} Ed, Springer, 2017. (§3.4.3)

‡D.C. Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.3.3, §3.10.1)

♠A.M. Legendre, *Nouvelles Méthodes pour la Détermination des Orbites des Comètes*, 1806.

♣Gauss, *Theoria Motus Corporum Coelestrium in Sectionibus Conicis Solem Ambientium*, 1809.

1-FACTOR UNBALANCED LINEAR MODEL (PREDICTED RESPONSES & RESIDUALS):

Given a 1-factor unbalanced linear model:

$$X_{ij} = \mu + \alpha_i^A + E_{ij} \quad \text{where} \quad E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$$

Then the corresponding **predicted responses**, denoted $\hat{x}_{ij}$, are:

$$\hat{x}_{ij} := \hat{\mu} + \hat{\alpha}_i^A = \overline{x}_{\bullet\bullet} + (\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet}) = \overline{x}_{i\bullet}$$

Moreover, the corresponding **residuals**, denoted x_{ij}^{res}, are:

$$x_{ij}^{res} := x_{ij} - \hat{x}_{ij} = x_{ij} - \overline{x}_{i\bullet}$$

1-FACTOR LINEAR MODEL (GAUSS[1]-MARKOV[2] THEOREM):

Given a 1-factor unbalanced linear model: $X_{ij} = \mu + \alpha_i^A + E_{ij}$

Moreover, suppose the following conditions are all satisfied:

$$\begin{aligned} \mathbb{E}[E_{ij}] &= 0 && \text{(errors are all centered at zero)} \\ \mathbb{V}[E_{ij}] &= \sigma^2 && \text{(errors all have the same finite variance)} \\ \mathbb{C}[E_{ij}, E_{i'j'}] &= 0 && \text{(errors are uncorrelated when } i \neq i' \text{ or } j \neq j'\text{)} \end{aligned}$$

Then, the least-squares estimators $\hat{\mu}, \hat{\alpha}_i^A$ are all BLUE's.

[1]C.F. Gauss, "Theoria Combinationis Observationum Erroribus Minimis Obnoxiae", (1823), 1-58.

[2]A.A. Markov, *Calculus of Probabilities*, 1^{st} Edition, 1900.

1-FACTOR UNBALANCED COMPLETELY RANDOMIZED ANOVA (1F ucrANOVA) [DEVORE 10.3]

- **1F ucrANOVA (MOTIVATION):** A 1F ucrANOVA is used if:
  - Some experimental units (EU's) in a balanced exp. malfunction, bite experimenters†, move away or die.
  - The levels of Factor A naturally differ in size – e.g. classroom rosters†.
  - Some levels of Factor A are prohibitively expensive to carry out‡ (and, hence, have fewer EU's).
  - Some levels of Factor A are far more interesting than others‡ (and, hence, have more EU's).

  †D.C. Howell, *Statistical Methods for Psychology*, 7^{th} Edition, Cengage, 2010. (§15.2)
  ‡D.C. Montgomery, *Design and Analysis of Experiments*, 7^{th} Edition, Wiley, 2009. (§15.2)

- **1F ucrANOVA (FIXED EFFECTS MODEL ASSUMPTIONS):**
  - ⋆ (**1 Desired Factor**) Factor A has I levels.
  - ⋆ (**All Factor Levels are Considered**) AKA Fixed Effects.
  - ⋆ (**Replication in Groups**) Each group has $J_i > 1$ units.
  - ⋆ (**Distinct Exp. Units**) All $\sum_i J_i$ units are distinct from each other.
  - ⋆ (**Random Assignment across Groups**)
  - ⋆ (**Independence**) All measurements on units are independent.
  - ⋆ (**Normality**) All groups are approximately normally distributed.
  - ⋆ (**Equal Variances**) All groups have approximately same variance.

  Mnemonic: **1DF AFLaC RiG DEU | RAaG | I.N.EV**

- **1F ucrANOVA (SUMS OF SQUARES "PARTITION" VARIATION):** $(n := \sum_i J_i)$

$$\underbrace{\mathrm{SS}_{total}}_{Total\ Variation\ in\ Experiment} = \underbrace{\mathrm{SS}_A}_{Variation\ due\ to\ Factor\ A} + \underbrace{\mathrm{SS}_{res}}_{Unexplained\ Variation}$$

$$\sum_{ij}(x_{ij} - \hat{\mu})^2 = \sum_{ij}(\hat{\alpha}_i^A)^2 + \sum_{ij}(x_{ij}^{res})^2$$

$$\sum_i \sum_{j=1}^{J_i}(x_{ij} - \overline{x}_{\bullet\bullet})^2 = \sum_i \sum_{j=1}^{J_i}(\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet})^2 + \sum_i \sum_{j=1}^{J_i}(x_{ij} - \overline{x}_{i\bullet})^2$$

$$\underbrace{\nu}_{Total\ dof's\ in\ Experiment} = \underbrace{\nu_A}_{'Between\ Groups'\ dof's} + \underbrace{\nu_{res}}_{'Within\ Groups'\ dof's}$$

$$\nu = n - 1 \qquad \nu_A = I - 1 \qquad \nu_{res} = n - I$$

- **1F ucrANOVA (EXPECTED MEAN SQUARES):**

$$(i)\ \mathbb{E}[\mathrm{MS}_{res}] = \sigma^2, \qquad (ii)\ \mathbb{E}[\mathrm{MS}_A] = \sigma^2 + \frac{1}{I-1}\sum_i J_i(\alpha_i^A)^2$$

- **1F ucrANOVA (POINT ESTIMATORS OF σ^2):**
  - (i) MS_{res} is always an unbiased point estimator of the population variance: $\mathbb{E}[\mathrm{MS}_{res}] = \sigma^2$
  - (ii) If the status quo prevails, MS_A is an unbiased estimator of σ^2: H_0 is indeed true $\implies \mathbb{E}[\mathrm{MS}_A] = \sigma^2$
  - (iii) If the status quo fails, MS_A tends to overestimate σ^2: H_0 is indeed false $\implies \mathbb{E}[\mathrm{MS}_A] > \sigma^2$

1-FACTOR UNBALANCED COMPLETELY RANDOMIZED ANOVA (1F ucrANOVA) [DEVORE 10.3]

- **1F ucrANOVA (FIXED EFFECTS LINEAR MODEL):**

1F ucrANOVA Fixed Effects Linear Model		
I	$\equiv$	# groups to compare
J_i	$\equiv$	# measurements in i^{th} group
X_{ij}	$\equiv$	rv for j^{th} measurement taken from i^{th} group
μ_i	$\equiv$	Mean of i^{th} population or true average response from i^{th} group
μ	$\equiv$	Common population mean or true average overall response
α_i^A	$\equiv$	Deviation from μ due to i^{th} group
E_{ij}	$\equiv$	Deviation from μ due to random error
ASSUMPTIONS: $E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$		
$X_{ij} = \mu + \alpha_i^A + E_{ij}$ where $\sum_i J_i \alpha_i^A = 0$		
$H_0^A:$ All $\alpha_i^A = 0$ $H_A^A:$ Some $\alpha_i^A \neq 0$		

- **1F ucrANOVA (PROCEDURE):**

1. Determine df's: $n := \sum_i J_i$, $\nu_A = I - 1$, $\nu_{res} = n - I$
2. Compute Group Means (if not provided): $\overline{x}_{i\bullet} := \underbrace{\frac{1}{J_i}\sum_{j=1}^{J_i} x_{ij}}_{\text{Given observations}}$ for $i = 1, \cdots, I$
3. Compute Group Variances (if not provided): $s_i^2 := \underbrace{\frac{1}{J_i - 1}\sum_{j=1}^{J_i}(x_{ij} - \overline{x}_{i\bullet})^2}_{\text{Given observations}} = \underbrace{\sqrt{J_i} \cdot \widehat{\sigma}_{\overline{x}_{i\bullet}}}_{\text{Given ESE's}}$ for $i = 1, \cdots, I$
4. Compute Grand Mean: $\overline{x}_{\bullet\bullet} := \frac{1}{I}\sum_i \overline{x}_{i\bullet}$
5. Compute $\text{SS}_{res} := \sum_i \sum_{j=1}^{J_i}(x_{ij}^{res})^2 = \sum_i \sum_{j=1}^{J_i}(x_{ij} - \overline{x}_{i\bullet})^2 = \sum_i (J_i - 1) \cdot s_i^2$
6. Compute $\text{SS}_A := \sum_i \sum_{j=1}^{J_i}(\hat{\alpha}_i^A)^2 = \sum_i \sum_{j=1}^{J_i}(\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet})^2$
7. Compute Mean Squares: $\text{MS}_{res} := \dfrac{\text{SS}_{res}}{\nu_{res}}$, $\text{MS}_A := \dfrac{\text{SS}_A}{\nu_A}$
8. Compute Test Statistic Value: $f_A = \dfrac{\text{MS}_A}{\text{MS}_{res}}$
9. Compute P-value: $p_A := \mathbb{P}(F > f_A) \approx 1 - \Phi_F(f_A; \nu_A, \nu_{res})$
10. Render Decision:
    - (by software) If $p_A \leq \alpha$ then reject H_0^A in favor of H_A^A, else accept H_0^A.
    - (by hand) If $f_A \geq f^*_{\nu_A, \nu_{res}; \alpha}$ then reject H_0^A in favor of H_A^A, else accept H_0^A.

- **1F ucrANOVA (TABLE):**

1F ucrANOVA Table (Significance Level α)

Variation Source	df	Sum of Squares	Mean Square	F Stat Value	P-value	Decision
Factor A	ν_A	SS_A	MS_A	f_A	p_A	Accept/Reject H_0^A
Unknown	ν_{res}	SS_{res}	MS_{res}			
Total	ν	SS_{total}				

1F ucrANOVA (TUKEY-KRAMER COMPARISONS) [DEVORE 10.3]

- **SIMULTANEOUS Q-CI's FOR MEAN DIFFERENCES:**

Given an unbalanced experiment with I groups each of size J_i such that the 1F ucrANOVA assumptions are satisfied.

Then the approximate simultaneous $100(1-\alpha)\%$ Q-CI's for all mean differences $\mu_i - \mu_j$ are:

$$(\overline{x}_{i\bullet} - \overline{x}_{j\bullet}) \pm q^*_{I,\nu_{res};\alpha} \cdot \sqrt{\text{MS}_{res} \cdot \frac{1}{2}\left(\frac{1}{J_i} + \frac{1}{J_j}\right)} \qquad \forall i < j \qquad (n := \textstyle\sum_i J_i, \qquad \nu_{res} := n - I)$$

If Q-CI for $\mu_i - \mu_j$ does not contain zero, then μ_i & μ_j significantly differ.

- **TUKEY-KRAMER COMPLETE PAIRWISE POST-HOC COMPARISON:** (Simpler than finding Q-CI's)

Given an unbalanced experiment with I groups each of size J_i $(n := \sum_i J_i)$
where 1F ucrANOVA rejects H_0^A at significance level α and the J_i's only differ slightly.
Then, to determine which population means significantly differ:

1. Sort the group means in ascending order: $\overline{x}_{(1)\bullet} \leq \overline{x}_{(2)\bullet} \leq \cdots \leq \overline{x}_{(I)\bullet}$
2. Compute significant difference widths $w_{(ij)} = q^*_{I,\nu_{res};\alpha} \cdot \sqrt{\text{MS}_{res} \cdot \frac{1}{2}\left(\frac{1}{J_{(i)}} + \frac{1}{J_{(j)}}\right)}$ $(\nu_{res} := n - I)$
3. If $\overline{x}_{(j)\bullet} \in \left[\overline{x}_{(i)\bullet},\ \overline{x}_{(i)\bullet} + w_{(ij)}\right]$, underline $\overline{x}_{(i)\bullet}$ and $\overline{x}_{(j)\bullet}$ with new line.
4. Repeat STEP 1 with all sorted mean pairs $\overline{x}_{(i)\bullet}, \overline{x}_{(j)\bullet}$ such that $i < j$.

Interpretation:

- Group means sharing a common underline implies they are not significantly different from one another.
- Group means not sharing a common underline implies they are significantly different from one another.

- **1F ANOVA MODEL CHECKING: STANDARDIZED RESIDUALS:**

Given a 1-factor experiment, either balanced or only slightly unbalanced:

$$X_{ij} = \mu + \alpha_i^A + E_{ij}$$

Moreover, suppose 1F bcrANOVA / ucrANOVA was performed accordingly.

Then, the **standardized residuals**† are defined to be:

$$z_{ij}^{res} := \frac{x_{ij}^{res}}{\sqrt{\text{SS}_{res}/(n-1)}}$$

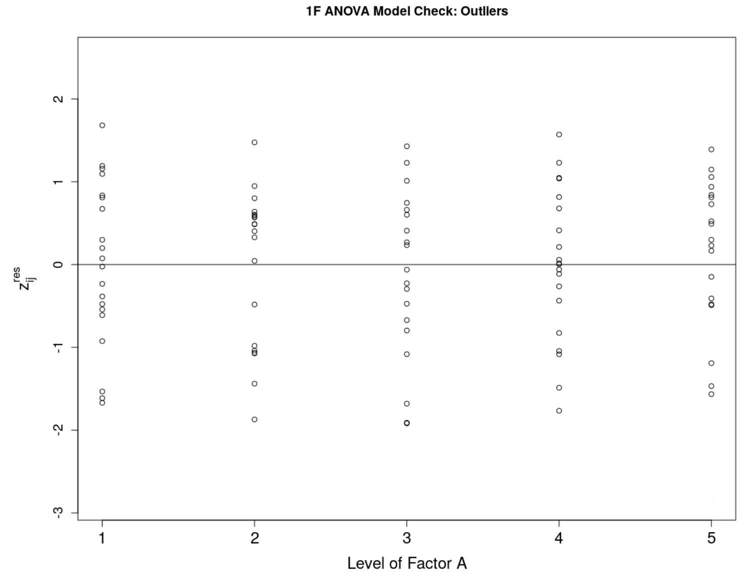
An alternative definition‡ that's reasonable but not used here is: $\dfrac{x_{ij}^{res}}{\sqrt{\text{MS}_{res}}}$

†Dean, Voss *et al*, *Design & Analysis of Experiments*, 2^{nd} Ed, 2017. (§5.2.1)

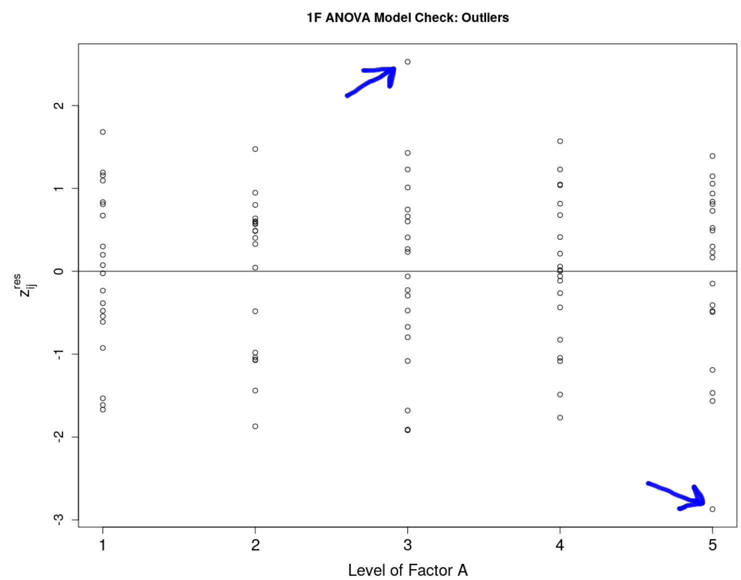
‡Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.4.1)

1F ANOVA MODEL CHECK PLOTS (OUTLIERS)

- **GOOD PLOT SUGGESTING NO OUTLIERS ARE PRESENT:**



- **BAD PLOT SUGGESTING THE PRESENCE OF (POSSIBLE) OUTLIERS:**



Measurements between two and three standard deviations are possibly outliers.

Measurements beyond three standard deviations are definitely outliers.

- **MITIGATION WHEN OUTLIER(S) ARE PRESENT:**

If outlier was due to measurement error, correct it†‡.

Else, it may be due to violation(s) of the ANOVA assumptions†.

Else, the 1-factor linear model may be insufficient†.

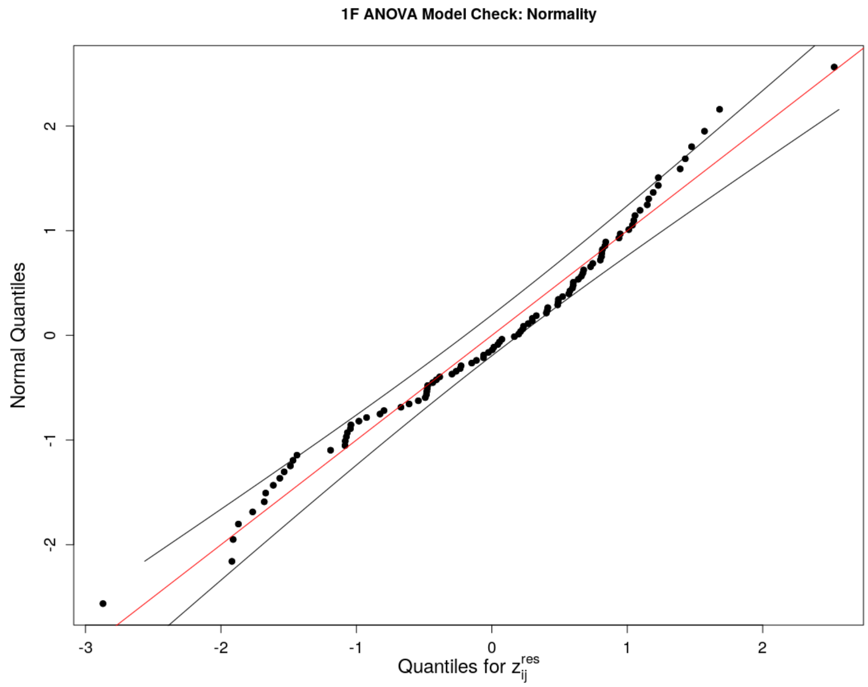
"We should be careful not to reject or discard an outlying observation unless we have reasonably non-statistical grounds for doing so. At worst, you may end up with two analyses; one with the outlier and one without."‡

†A. Dean, D. Voss, D. Draguljić, *Design & Analysis of Experiments*, 2^{nd} Ed, 2017. (§5.4)

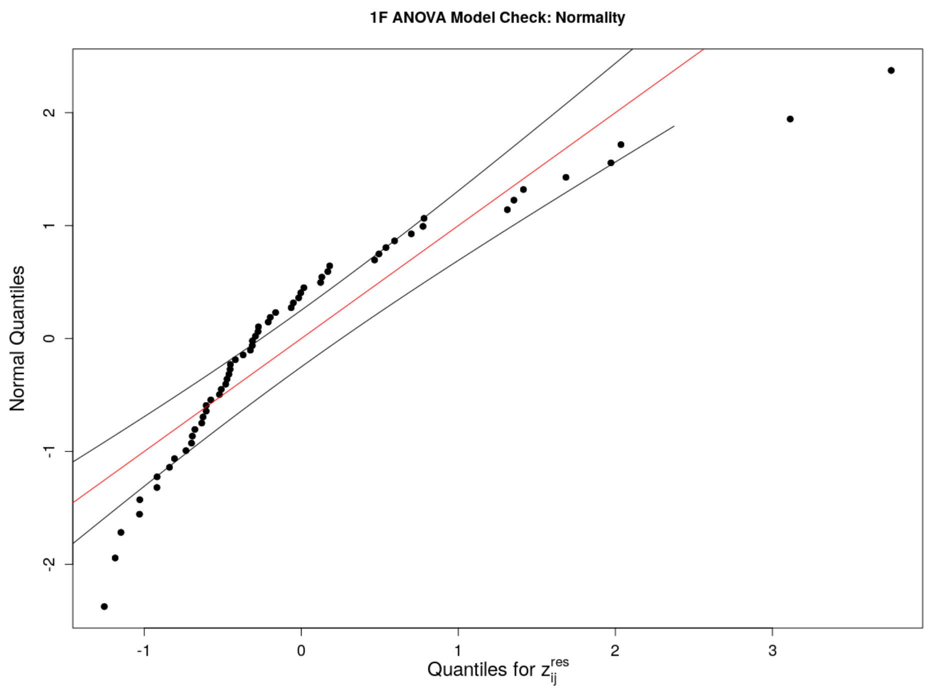
‡D.C. Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.4.1)

1F ANOVA MODEL CHECK PLOTS (NORMALITY)

- **GOOD PLOT SUGGESTING NORMALITY ASSUMPTION IS SATISFIED:**



- **BAD PLOT SUGGESTING NORMALITY ASSUMPTION IS VIOLATED:**



- **MITIGATION WHEN NORMALITY ASSUMPITION IS VIOLATED:**

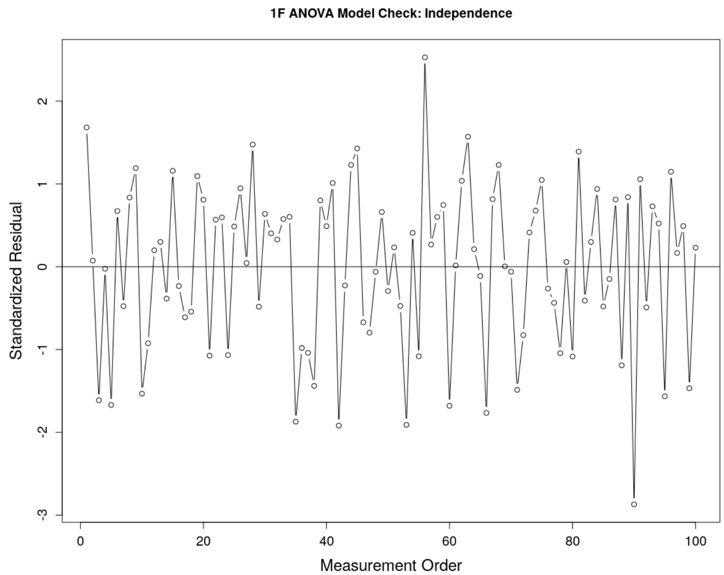
  Q: How to perform a 1F ANOVA when the Normality Assumption is violated?

  A: Perform a 1F Kruskal-Wallis♠ ANOVA which does not assume normality – to be covered in Ch15.

♠W.H. Kruskal, W.A. Wallis, "Use of Ranks in 1-Criterion Variance Analysis", *J. Amer. Stat. Assoc.*, **47** (1952), 583-621.

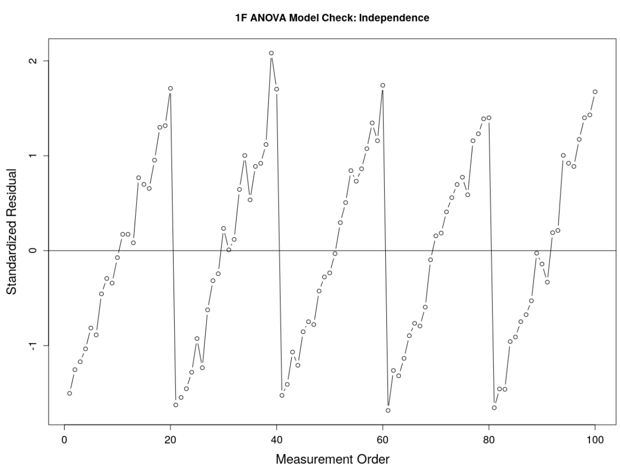
1F ANOVA MODEL CHECK PLOTS (INDEPENDENCE)

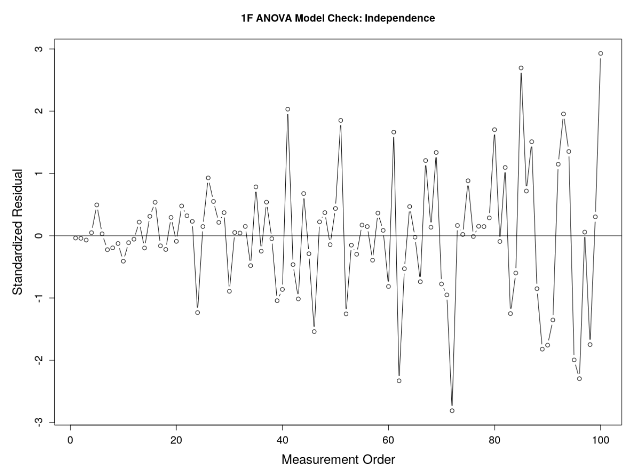
- **GOOD PLOT SUGGESTING INDEPENDENCE ASSUMPTION IS SATISFIED:**



There's no discernible pattern.

- **BAD PLOTS SUGGESTING INDEPENDENCE ASSUMPTION IS VIOLATED:**





There's a clear pattern in each plot: (left) cycle and (right) fan

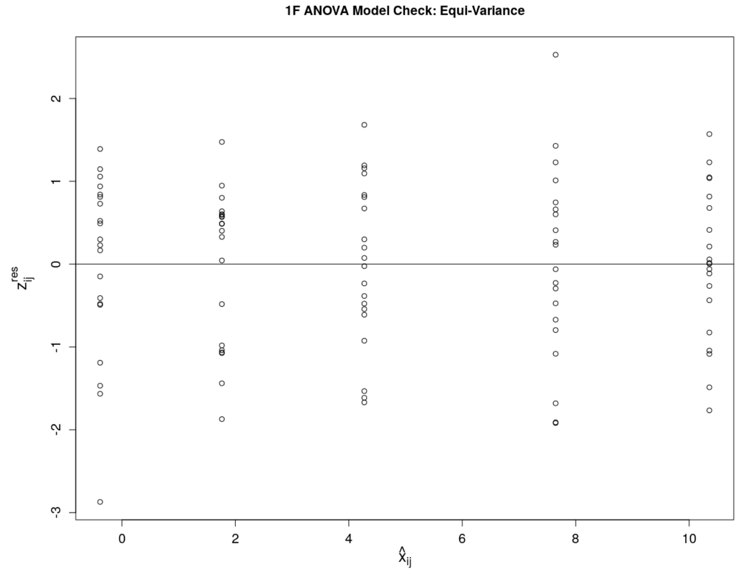
- **MITIGATION WHEN INDEPENDENCE ASSUMPITION IS VIOLATED:**
  - If randomization was not used, redo the experiment using randomization‡.
  - If randomization was used, then use a more complicated model†:
    - 2-Factor ANOVA – to be covered in Ch11
    - Analysis of Covariance (ANCOVA) – beyond scope of this course

†A. Dean, D. Voss, D. Draguljić, *Design & Analysis of Experiments*, 2^{nd} Ed, Springer, 2017. (§5.5)

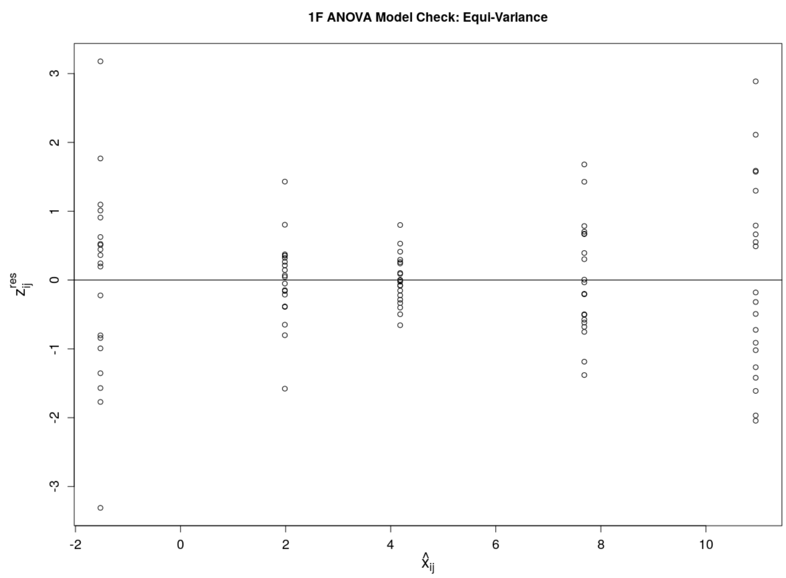
‡D.C. Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.4.2)

1F ANOVA MODEL CHECK PLOTS (EQUI-VARIANCE)

- **GOOD PLOT SUGGESTING EQUI-VARIANCE ASSUMPTION IS SATISFIED:**



- **BAD PLOT SUGGESTING EQUI-VARIANCE ASSUMPTION IS VIOLATED:**



- **MITIGATION WHEN EQUI-VARIANCE ASSUMPITION IS VIOLATED:**

  Perform the appropriate **variance-stabilizing data transformation**†‡♣ from the following:

  $\log X,\ \log(1+X),\ \log(1+\min x_{ij}+X),\ \sqrt{X},\ \sqrt{0.5+X},\ \sqrt{X}+\sqrt{1+X},\ 1/X,\ 1/\sqrt{X},\ \arcsin(\sqrt{X}),\ 2\arcsin(\sqrt{X \pm 1/2m})$

  If data are counts or Poisson-like, use a square-root transformation†‡♣.

  If data are proportions or Binomial-like, use an arcsine transformation†♣.

  When in doubt, plot $\log s_i$ vs. $\log(\overline{x}_{i\bullet})$ to help determine appropriate data transformation†‡.

  If data transformations do not seem to help much, a more robust method is necessary♡.

†A. Dean, D. Voss, D. Draguljić, *Design & Analysis of Experiments*, 2^{nd} Ed, Springer, 2017. (§5.6.2)

‡D.C. Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.4.3)

♣D.C. Howell, *Statistical Methods for Psychology*, 7^{th} Ed, Cengage, 2010. (§11.9)

♡R.J. Grissom, "Heterogeneity of Variance in Clinical Data", *J. Consulting & Clinical Psychology*, **68** (2000), 155-165.

EX 10.3.1: The lifetimes of three light bulb brands were measured:

BULB BRAND	**(BULB LIFETIMES in yrs)**
Brand 1 ($x_{1\bullet}$)	9.22, 9.07, 8.95, 8.98, 9.54
Brand 2 ($x_{2\bullet}$)	8.92, 8.88, 9.10
Brand 3 ($x_{3\bullet}$)	9.08, 8.99, 9.06, 8.93

(a) Formulate this experiment as a 1-Factor ANOVA fixed effects linear model.

(b) Perform the appropriate 1-Factor ANOVA at significance level $\alpha = 0.01$ – compute both the F-cutoff and P-value.

(c) Summarize everything in an 1-Factor ANOVA table.

EX 10.3.2: Dentists use resin composites and ceramic fillings among others for cavities in teeth. The shear bond strengths of resin composite-ceramic bonds formed from three possible configurations (conventional, all-composite, reversed) were measured (in MPa) and summarized in the following table:

GROUP:	SAMPLE SIZE:	MEAN:	STD DEV:
Conventional	9	$\overline{x}_{1\bullet} = 10.37$	$s_1 = 1.99$
All-Composite	8	$\overline{x}_{2\bullet} = 21.82$	$s_2 = 2.45$
Reversed	6	$\overline{x}_{3\bullet} = 18.02$	$s_3 = 2.52$

This table and all the details regarding the experiment can be found in the following paper:

A. Della Bona, R. van Noort, "Shear vs. Tensile Bond Strength of Resin Composite Bonded to Ceramic", *Journal of Dental Research*, **74** (1995), 1591-1596.

(a) Formulate this experiment as a 1-Factor ANOVA fixed effects linear model.

(b) Perform the appropriate 1-Factor ANOVA at significance level $\alpha = 0.05$ – compute both the F-cutoff and P-value.

(c) Perform the appropriate Tukey-Kramer Complete Pairwise Post-Hoc Comparison.