EX 11.1.1: The lifetimes of three light bulb brands were blocked by raw material batch and then measured:

BULB LIFETIME (in years)						
BLOCK B: → **FACTOR A:** ↓	Batch 1 ($x_{\bullet 1}$)	Batch 2 ($x_{\bullet 2}$)	Batch 3 ($x_{\bullet 3}$)	Batch 4 ($x_{\bullet 4}$)	Batch 5 ($x_{\bullet 5}$)	**TOTAL** ($\sum_j x_{ij}$)
Brand 1 ($x_{1\bullet}$)	9.22	9.07	8.95	8.98	9.54	**45.76**
Brand 2 ($x_{2\bullet}$)	8.92	8.88	9.10	8.71	8.85	**44.46**
Brand 3 ($x_{3\bullet}$)	9.08	8.99	9.06	8.93	9.02	**45.08**
TOTAL ($\sum_i x_{ij}$)	**27.22**	**26.94**	**27.11**	**26.62**	**27.41**	$\sum_i \sum_j x_{ij} = \boxed{\mathbf{135.30}}$

(a) Formulate this experiment as a 2-Factor fixed effects linear model. In this context, what does "fixed effects" assume?

$$X_{ij} = \mu + \alpha_i^A + \alpha_j^{[B]} + E_{ij} \quad \text{where} \quad \begin{array}{ccl} \mu & \equiv & \text{Grand average bulb lifetime over all 3 brands and all 5 batches} \\ \alpha_i^A & \equiv & \text{Bulb lifetime deviation from } \mu \text{ due to Brand } i \\ \alpha_j^{[B]} & \equiv & \text{Bulb lifetime deviation from } \mu \text{ due to Batch } j \\ E_{ij} & \equiv & \text{Bulb lifetime deviation from } \mu \text{ due to random error/noise} \end{array}$$

Here, "fixed effects" assumes that <u>all available</u> bulb brands and batches are considered.

i.e. There is no Brand 4, Brand 5, etc... Similarly, there is no Batch 6, Batch 7, etc...

(b) State the appropriate null hypothesis H_0^A and alternative hypothesis H_A^A.

$$\begin{array}{ll} H_0^A: & \alpha_1^A = \alpha_2^A = \alpha_3^A = 0 \\ H_A^A: & \text{Some } \alpha_i^A \neq 0 \end{array} \qquad \text{OR EQUIVALENTLY} \qquad \begin{array}{ll} H_0^A: & \text{All } \alpha_i^A = 0 \\ H_A^A: & \text{Some } \alpha_i^A \neq 0 \end{array}$$

(c) Perform a 2-Factor Randomized Complete Block ANOVA (2F rcbANOVA) with ($\alpha = 0.01$) significance level.

Was the chosen blocking effective? To save time and tedium: $\text{SS}_{total} = 0.4946$, $\text{SS}_{res} \approx 0.20595$

1^{st}, determine relevant counts: $I \equiv (\#\text{ levels of Factor A}) = 3$, $J \equiv (\#\text{ levels of Blocking Factor B}) = 5$

2^{nd}, compute degrees of freedom: $\nu_A := I - 1 = 2$, $\nu_{[B]} := J - 1 = 4$, $\nu_{res} := (I-1)(J-1) = 8$

3^{rd}, compute group means: $\overline{x}_{i\bullet} := \frac{1}{J}\sum_j x_{ij}$ $\qquad \overline{x}_{\bullet j} := \frac{1}{I}\sum_i x_{ij}$

$\overline{x}_{1\bullet} = \frac{1}{5}(9.22 + 9.07 + 8.95 + 8.98 + 9.54) = \frac{1}{5} \cdot \mathbf{45.76} = 9.152$ Similarly: $\overline{x}_{2\bullet} = 8.892$, $\overline{x}_{3\bullet} = 9.016$

$\overline{x}_{\bullet 1} = \frac{1}{3}(9.22 + 8.92 + 9.08) = \frac{1}{3} \cdot \mathbf{27.22} \approx 9.0733$, $\overline{x}_{\bullet 2} = 8.9800$, $\overline{x}_{\bullet 3} \approx 9.0367$, $\overline{x}_{\bullet 4} \approx 8.8733$, $\overline{x}_{\bullet 5} \approx 9.1367$

4^{th}, compute grand mean: $\overline{x}_{\bullet\bullet} := \frac{1}{IJ}\sum_{ij} x_{ij} = \frac{1}{IJ}\sum_i \sum_j x_{ij} = \frac{1}{3 \cdot 5} \cdot \mathbf{135.30} = 9.02$

5^{th}, compute the two unknown sums of squares: (SS_{total} & SS_{res} are given above.)

$$\text{SS}_A := \sum_{ij}(\hat{\alpha}_i^A)^2 = J \cdot \sum_i (\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet})^2 = 5 \cdot [(9.152 - 9.02)^2 + (8.892 - 9.02)^2 + (9.016 - 9.02)^2] \approx 0.16912$$

$$\text{SS}_{[B]} := \sum_{ij}(\hat{\alpha}_j^{[B]})^2 = I \cdot \sum_j (\overline{x}_{\bullet j} - \overline{x}_{\bullet\bullet})^2 = 3 \cdot \left[\begin{array}{l} (9.0733 - 9.02)^2 + (8.98 - 9.02)^2 + (9.0367 - 9.02)^2 \\ + (8.8733 - 9.02)^2 + (9.1367 - 9.02)^2 \end{array}\right] \approx 0.11958$$

6^{th}, compute mean squares: $\text{MS}_A := \frac{\text{SS}_A}{\nu_A} = 0.084560$, $\text{MS}_{[B]} := \frac{\text{SS}_{[B]}}{\nu_{[B]}} \approx 0.029895$, $\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}} = 0.025744$

7^{th}, compute F-test statistic values: $f_A := \frac{\text{MS}_A}{\text{MS}_{res}} = \frac{0.084560}{0.025744} \approx 3.285$, $f_{[B]} := \frac{\text{MS}_{[B]}}{\text{MS}_{res}} = \frac{0.029895}{0.025744} \approx 1.161$

8^{th}, lookup F-cutoffs in F table (§9.5): $f^*_{\nu_A,\nu_{res};\alpha} = f^*_{2,8;0.01} \overset{LOOKUP}{\approx} 8.649$, $f^*_{\nu_{[B]},\nu_{res};\alpha} = f^*_{4,8;0.01} \overset{LOOKUP}{\approx} 7.006$

9^{th}, render appropriate decisions:

Since	f_A	$\approx$	$3.285 < 8.649$	$\approx$	$f^*_{\nu_A,\nu_{res};\alpha}$,	<u>accept</u> H_0^A, meaning bulb brand has no significant effect on lifetime.
Since	$f_{[B]}$	$\approx$	$1.161 < 7.006$	$\approx$	$f^*_{\nu_{[B]},\nu_{res};\alpha}$,	the chosen blocking on bulb batch was <u>not effective</u>.

(d) Compute & interpret the eta-squared and partial eta-squared effect size measures: $\hat{\eta}^2_A, \hat{\eta}^2_{[B]}$; $\hat{\eta}^2_{(A)}, \hat{\eta}^2_{([B])}$

$\hat{\eta}^2_A := \frac{\text{SS}_A}{\text{SS}_A + \text{SS}_{[B]} + \text{SS}_{res}} = \frac{0.16912}{0.16912 + 0.11958 + 0.20595} \approx \boxed{0.342} \implies$ 34.2% of the variation in bulb lifetime is due to bulb brand.

$\hat{\eta}^2_{[B]} := \frac{\text{SS}_{[B]}}{\text{SS}_A + \text{SS}_{[B]} + \text{SS}_{res}} = \frac{0.11958}{0.16912 + 0.11958 + 0.20595} \approx \boxed{0.242} \implies$ 24.2% of the variation in bulb lifetime is due to bulb batch.

$\hat{\eta}^2_{(A)} := \frac{\text{SS}_A}{\text{SS}_A + \text{SS}_{res}} = \frac{0.16912}{0.16912 + 0.20595} \approx \boxed{0.451} \implies$ 45.1% of the variation possibly due bulb brand is truly due to it.

$\hat{\eta}^2_{([B])} := \frac{\text{SS}_{[B]}}{\text{SS}_{[B]} + \text{SS}_{res}} = \frac{0.11958}{0.11958 + 0.20595} \approx \boxed{0.367} \implies$ 36.7% of the variation possibly due bulb batch is truly due to it.

EX 11.1.2: The counts of M&M's® peanut chocolate candies in seven equal-size bags are provided in this table†:

NUMBER OF CHOCOLATE CANDIES OF A GIVEN COLOR IN A GIVEN BAG								
BLOCK B: $\rightarrow$ **FACTOR A:** $\downarrow$	Bag 1 ($x_{\bullet 1}$)	Bag 2 ($x_{\bullet 2}$)	Bag 3 ($x_{\bullet 3}$)	Bag 4 ($x_{\bullet 4}$)	Bag 5 ($x_{\bullet 5}$)	Bag 6 ($x_{\bullet 6}$)	Bag 7 ($x_{\bullet 7}$)	**TOTAL** ($\sum_j x_{ij}$)
Blue ($x_{1\bullet}$)	8	7	5	7	6	8	6	**47**
Red ($x_{2\bullet}$)	2	2	5	3	5	4	5	**26**
Orange ($x_{3\bullet}$)	1	0	0	1	1	2	1	**6**
Green ($x_{4\bullet}$)	0	1	0	2	0	3	2	**8**
Brown ($x_{5\bullet}$)	5	6	6	7	5	7	5	**41**
Yellow ($x_{6\bullet}$)	2	1	3	1	2	3	1	**13**
TOTAL ($\sum_i x_{ij}$)	**18**	**17**	**19**	**21**	**19**	**27**	**20**	$\sum_i \sum_j x_{ij} = \boxed{\mathbf{141}}$

†This table is a simplified and modified version of the table (and experiment) found in:

T. Lin, M.S. Sanders, "A Sweet Way to Learn DoE", *Quality Progress*, **39** (2006), 88.

(a) Formulate this experiment as a 2-Factor fixed effects linear model.

$$X_{ij} = \mu + \alpha_i^A + \alpha_j^{[B]} + E_{ij} \quad \text{where} \quad \begin{array}{rcl} \mu & \equiv & \text{Grand average M\&M candy count over all 6 colors and all 7 bags} \\ \alpha_i^A & \equiv & \text{Candy count deviation from } \mu \text{ due to Color } i \\ \alpha_j^{[B]} & \equiv & \text{Candy count deviation from } \mu \text{ due to Bag } j \\ E_{ij} & \equiv & \text{Candy count deviation from } \mu \text{ due to random error/noise} \end{array}$$

(b) State the appropriate null hypothesis H_0^A and alternative hypothesis H_A^A.

$$\begin{array}{ll} H_0^A: & \alpha_1^A = \alpha_2^A = \alpha_3^A = \alpha_4^A = \alpha_5^A = \alpha_6^A = 0 \\ H_A^A: & \text{Some } \alpha_i^A \neq 0 \end{array} \qquad \text{OR EQUIVALENTLY} \qquad \begin{array}{ll} H_0^A: & \text{All } \alpha_i^A = 0 \\ H_A^A: & \text{Some } \alpha_i^A \neq 0 \end{array}$$

(c) Perform a 2-Factor Randomized Complete Block ANOVA (2F rcbANOVA) with ($\alpha = 0.05$) significance level.

Was the blocking effective? To save time: $SS_{total} \approx 257.643$, $SS_A \approx 217.357$, $SS_{[B]} \approx 10.810$, $SS_{res} \approx 29.476$

1st, determine relevant counts: $I \equiv (\#\text{ levels of Factor A}) = 6$, $J \equiv (\#\text{ levels of Blocking Factor B}) = 7$

2nd, compute degrees of freedom: $\nu_A := I - 1 = 5$, $\nu_{[B]} := J - 1 = 6$, $\nu_{res} := (I-1)(J-1) = 30$

3rd, compute mean squares: $MS_A := \frac{SS_A}{\nu_A} = 43.4714$, $MS_{[B]} := \frac{SS_{[B]}}{\nu_{[B]}} \approx 1.8017$, $MS_{res} := \frac{SS_{res}}{\nu_{res}} \approx 0.9825$

4th, compute F-test statistic values: $f_A := \frac{MS_A}{MS_{res}} = \frac{43.4714}{0.9825} \approx 44.2457$, $f_{[B]} := \frac{MS_{[B]}}{MS_{res}} = \frac{1.8017}{0.9825} \approx 1.8338$

5th, lookup F-cutoffs in F table (§9.5): $f^*_{\nu_A,\nu_{res};\alpha} = f^*_{5,30;0.05} \overset{LOOKUP}{\approx} 2.534$, $f^*_{\nu_{[B]},\nu_{res};\alpha} = f^*_{6,30;0.05} \overset{LOOKUP}{\approx} 2.421$

6th, render appropriate decisions:

$f_A \approx 44.2457 > 2.534 \approx f^*_{\nu_A,\nu_{res};\alpha}$, so reject H_0^A, so at least two candy color counts significantly differ in a bag.
$f_{[B]} \approx 1.8338 < 2.421 \approx f^*_{\nu_{[B]},\nu_{res};\alpha}$, so the chosen blocking on candy bag was barely not effective.

(d) Compute & interpret the eta-squared and partial eta-squared effect size measures: $\hat{\eta}_A^2$; $\hat{\eta}_{(A)}^2$

$$\hat{\eta}_A^2 := \frac{SS_A}{SS_A + SS_{[B]} + SS_{res}} = \frac{217.357}{217.357 + 10.810 + 29.476} \approx \boxed{0.844} \implies \text{84.4\% of the variation in candy count is due to candy color.}$$

$$\hat{\eta}_{(A)}^2 := \frac{SS_A}{SS_A + SS_{res}} = \frac{217.357}{217.357 + 29.476} \approx \boxed{0.881} \implies \text{88.1\% of the variation possibly due candy color is truly due to it.}$$

(e) Perform the appropriate Tukey Complete Pairwise Post-Hoc Comparison.

$$w = q^*_{I,\nu_{res};\alpha} \cdot \sqrt{MS_{res}/J} \approx q^*_{6,30;0.05} \cdot \sqrt{0.9825/7} \overset{LOOKUP}{\approx} \mathbf{4.301} \cdot \sqrt{0.9825/7} \approx 1.611 \qquad (\overline{x}_{1\bullet} = 47/7 \approx 6.714)$$

$\overline{x}_{1\bullet}$	$\overline{x}_{2\bullet}$	$\overline{x}_{3\bullet}$	$\overline{x}_{4\bullet}$	$\overline{x}_{5\bullet}$	$\overline{x}_{6\bullet}$
6.714	3.714	0.857	1.143	5.857	1.857

$\overset{SORT}{\Longrightarrow}$

$\overline{x}_{(1)\bullet}$	$\overline{x}_{(2)\bullet}$	$\overline{x}_{(3)\bullet}$	$\overline{x}_{(4)\bullet}$	$\overline{x}_{(5)\bullet}$	$\overline{x}_{(6)\bullet}$
$\overline{x}_{3\bullet}$	$\overline{x}_{4\bullet}$	$\overline{x}_{6\bullet}$	$\overline{x}_{2\bullet}$	$\overline{x}_{5\bullet}$	$\overline{x}_{1\bullet}$
0.857	1.143	1.857	3.714	5.857	6.714

(Underline means within $w \approx 1.611$ of each other.)

$\therefore$ In a typical bag... $\left\{ \begin{array}{l} \text{...there are significantly more blue \& brown M\&M's than red ones.} \\ \text{...there are significantly more red M\&M's than orange, green \& yellow ones.} \end{array} \right\}$