EX 11.2.1: The lifetimes of 24 light bulbs, available in two brands and three wattages, were randomized and then measured:

BULB LIFETIME (in years)				
WATTAGE: → **BRAND:** ↓	60-Watt ($x_{\bullet 1}$)	75-Watt ($x_{\bullet 2}$)	100-Watt ($x_{\bullet 3}$)	**TOTAL** ($\sum_j \sum_k x_{ijk}$)
Brand 1 ($x_{1\bullet}$)	9.23, 7.64, 8.59, 7.66	8.54, 5.98, 8.15, 8.30	1.29, 3.13, 1.42, 3.28	**73.21**
Brand 2 ($x_{2\bullet}$)	14.54, 13.77, 15.43, 14.20	10.82, 10.84, 12.86, 13.81	9.65, 9.00, 8.24, 8.61	**141.77**
TOTAL ($\sum_i \sum_k x_{ijk}$)	**91.06**	**79.30**	**44.62**	$\sum_i \sum_j \sum_k x_{ijk} =$ **214.98**

(a) Formulate this experiment as a 2-factor fixed effects linear model. In this context, what does "fixed effects" assume?

$X_{ijk} = \mu + \alpha_i^A + \alpha_j^{[B]} + \gamma_{ij}^{AB} + E_{ijk}$ where

$\mu \equiv$ Grand average bulb lifetime over all 2 brands and all 3 wattages

$(\alpha_i^A,\ \alpha_j^B) \equiv$ Bulb lifetime deviation from μ due to (Brand i, Wattage j)

$(\gamma_{ij}^{AB},\ E_{ijk}) \equiv$ Bulb lifetime deviation from μ due to (interaction, noise)

Here, "fixed effects" assumes all available brands and wattages are considered. (i.e. There is no Brand 3, 90-Watt, ...)

(b) State the appropriate null & alternative hypotheses for factor A, factor B and interaction AB.

$H_0^A: \quad \alpha_1^A = \alpha_2^A = 0 \qquad H_0^B: \quad \alpha_1^B = \alpha_2^B = \alpha_3^B = 0 \qquad H_0^{AB}: \quad \gamma_{11}^{AB} = \gamma_{12}^{AB} = \gamma_{13}^{AB} = \gamma_{21}^{AB} = \gamma_{22}^{AB} = \gamma_{23}^{AB} = 0$

$H_A^A: \quad \alpha_1^A \neq \alpha_2^A \qquad H_A^B: \quad \text{Some } \alpha_j^B \neq 0 \qquad H_A^{AB}: \quad \text{Some } \gamma_{ij}^{AB} \neq 0$

(c) Perform a 2-Factor Balanced Completely Randomized ANOVA (2F bcrANOVA) with ($\alpha = 0.01$) significance level.

To save time and tedium: $\text{SS}_{total} = 366.15840$, $\text{SS}_{res} = 18.75875$, (and utilize the row & column totals in the table!)

1^{st}, determine counts: $I \equiv (\#\text{ levels of Factor A}) = 2$, $J \equiv (\#\text{ levels of Factor B}) = 3$, $K \equiv (\text{Group size}) = 4$

2^{nd}, compute dof's: $\nu_A := I - 1 = 1$, $\nu_B := J - 1 = 2$, $\nu_{AB} := (I-1)(J-1) = 2$, $\nu_{res} := IJ(K-1) = 18$

3^{rd}, compute group means: $\overline{x}_{i\bullet\bullet} := \frac{1}{JK}\sum_j \sum_k x_{ijk}$ $\quad \overline{x}_{\bullet j\bullet} := \frac{1}{IK}\sum_i \sum_k x_{ijk}$ $\quad \overline{x}_{ij\bullet} := \frac{1}{K}\sum_k x_{ijk}$

$\overline{x}_{1\bullet} = \frac{\mathbf{73.21}}{3\cdot 4} \approx 6.1008$, $\overline{x}_{2\bullet} \approx 11.8142$; $\quad \overline{x}_{\bullet 1} = \frac{\mathbf{91.06}}{2\cdot 4} = 11.3825$, $\overline{x}_{\bullet 2} = 9.9125$, $\overline{x}_{\bullet 3} = 5.5775$

4^{th}, compute grand mean: $\overline{x}_{\bullet\bullet\bullet} := \frac{1}{IJK}\sum_{ijk} x_{ijk} = \frac{1}{IJK}\sum_i \sum_j \sum_k x_{ijk} = \frac{1}{2\cdot 3\cdot 4}\cdot \mathbf{214.98} = 8.9575$

5^{th}, compute the three unknown sums of squares: (SS_{total} & SS_{res} are given above.)

$\text{SS}_A := \sum_{ijk}(\hat{\alpha}_i^A)^2 = JK\cdot\sum_i(\overline{x}_{i\bullet\bullet} - \overline{x}_{\bullet\bullet\bullet})^2 = (3)(4)\cdot[(6.1008 - 8.9575)^2 + (11.8142 - 8.9575)^2] \approx 195.8576$

$\text{SS}_B := \sum_{ijk}(\hat{\alpha}_j^B)^2 = IK\cdot\sum_j(\overline{x}_{\bullet j\bullet} - \overline{x}_{\bullet\bullet\bullet})^2 = (2)(4)\cdot[(11.3825 - 8.9575)^2 + (9.9125 - 8.9575)^2 + (5.5775 - 8.9575)^2] \approx 145.8364$

$\text{SS}_{AB} = \text{SS}_{total} - \text{SS}_A - \text{SS}_B - \text{SS}_{res} = 366.15840 - 195.8576 - 145.8364 - 18.75875 \approx 5.80565$

6^{th}: $\text{MS}_A := \frac{\text{SS}_A}{\nu_A} \approx 195.8576$, $\text{MS}_B := \frac{\text{SS}_B}{\nu_B} \approx 72.9182$, $\text{MS}_{AB} := \frac{\text{SS}_{AB}}{\nu_{AB}} \approx 2.9028$, $\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}} \approx 1.0422$

7^{th}: $f_A := \frac{\text{MS}_A}{\text{MS}_{res}} = \frac{195.8576}{1.0422} \approx 187.927$, $f_B := \frac{\text{MS}_B}{\text{MS}_{res}} = \frac{72.9182}{1.0422} \approx 69.966$, $f_{AB} := \frac{\text{MS}_{AB}}{\text{MS}_{res}} = \frac{2.9028}{1.0422} \approx 2.785$

8^{th}: $f^*_{\nu_A,\nu_{res};\alpha} = f^*_{1,18;0.01} \overset{LOOKUP}{\approx} 8.285$, $f^*_{\nu_B,\nu_{res};\alpha} = f^*_{2,18;0.01} \overset{LOOKUP}{\approx} 6.013$, $f^*_{\nu_{AB},\nu_{res};\alpha} \overset{LOOKUP}{\approx} 6.013$

9^{th}, render appropriate decisions:

Since $f_A \approx 187.9 > 8.285 \approx f^*_{\nu_A,\nu_{res};\alpha}$, reject H_0^A, meaning bulb brand has a significant effect on lifetime.
Since $f_B \approx 70 > 6.013 \approx f^*_{\nu_B,\nu_{res};\alpha}$, reject H_0^B, meaning bulb wattage has a significant effect on lifetime.
Since $f_{AB} \approx 2.785 < 6.013 \approx f^*_{\nu_{AB},\nu_{res};\alpha}$, accept H_0^{AB}, meaning there is no significant interaction.

(d) Perform & interpret the appropriate Tukey Complete Pairwise Post-Hoc Comparison.

$w_A = q^*_{I,\nu_{res};\alpha}\cdot\sqrt{\text{MS}_{res}/(JK)} \approx 4.071\cdot 0.2947 \approx 1.20$, $\quad w_B = q^*_{J,\nu_{res};\alpha}\cdot\sqrt{\text{MS}_{res}/(IK)} \approx 4.702\cdot 0.3609 \approx 1.70$

$\overline{x}_{(1)\bullet}$	$\overline{x}_{(2)\bullet}$
$\overline{x}_{1\bullet}$	$\overline{x}_{2\bullet}$
6.1008	11.8142

(Underline sorted means $\overline{x}_{(i)\bullet}$ within $w_A \approx 1.20$ of each other, if any.)

$\overline{x}_{\bullet(1)}$	$\overline{x}_{\bullet(2)}$	$\overline{x}_{\bullet(3)}$
$\overline{x}_{\bullet 3}$	$\overline{x}_{\bullet 2}$	$\overline{x}_{\bullet 1}$
44.62	79.30	91.06

(Underline sorted means $\overline{x}_{\bullet(j)}$ within $w_B \approx 1.70$ of each other, if any.)

∴ Brand 2 bulbs have significantly longer lifetimes than Brand 1 bulbs.
Higher wattage bulbs have significantly shorter lifetimes.

EX 11.2.2: The lifetimes of 24 light bulbs, available in two brands and three wattages, were randomized and then measured:

BULB LIFETIME (in years)				
WATTAGE: → **BRAND:** ↓	60-Watt ($x_{\bullet 1}$)	75-Watt ($x_{\bullet 2}$)	100-Watt ($x_{\bullet 3}$)	**TOTAL** ($\sum_j \sum_k x_{ijk}$)
Brand 1 ($x_{1\bullet}$)	10.78, 9.87, 12.37, 8.38	5.79, 4.35, 7.02, 5.16	2.51, 2.70, 5.05, 2.46	**76.44**
Brand 2 ($x_{2\bullet}$)	11.31, 12.63, 11.60, 12.15	16.31, 14.33, 14.66, 15.19	7.73, 8.27, 7.20, 10.86	**142.24**
TOTAL ($\sum_i \sum_k x_{ijk}$)	**89.09**	**82.81**	**46.78**	$\sum_i \sum_j \sum_k x_{ijk} =$ **218.68**

(a) Formulate this experiment as a 2-factor fixed effects linear model.

$X_{ijk} = \mu + \alpha_i^A + \alpha_j^B + \gamma_{ij}^{AB} + E_{ijk}$ where

$\mu \equiv$ Grand average bulb lifetime over all 2 brands and all 3 wattages

$(\alpha_i^A,\ \alpha_j^B) \equiv$ Bulb lifetime deviation from μ due to (Brand i, Wattage j)

$(\gamma_{ij}^{AB},\ E_{ijk}) \equiv$ Bulb lifetime deviation from μ due to (interaction, noise)

(b) State the appropriate null & alternative hypotheses for factor A, factor B and interaction AB.

$H_0^A: \ \alpha_1^A = \alpha_2^A = 0$ $\qquad H_0^B: \ \alpha_1^B = \alpha_2^B = \alpha_3^B = 0$ $\qquad H_0^{AB}: \ \gamma_{11}^{AB} = \gamma_{12}^{AB} = \gamma_{13}^{AB} = \gamma_{21}^{AB} = \gamma_{22}^{AB} = \gamma_{23}^{AB} = 0$

$H_A^A: \ \alpha_1^A \neq \alpha_2^A$ $\qquad H_A^B: \ \text{Some } \alpha_j^B \neq 0$ $\qquad H_A^{AB}: \ \text{Some } \gamma_{ij}^{AB} \neq 0$

(c) Perform a 2-Factor Balanced Completely Randomized ANOVA (2F bcrANOVA) with ($\alpha = 0.05$) significance level.

To save time: $SS_{total} = 402.38170$, $SS_A \approx 180.40167$, $SS_B \approx 130.32231$, $SS_{AB} \approx 63.58691$, $SS_{res} = 28.07085$

1^{st}, determine counts: $I \equiv (\#\text{ levels of Factor A}) = 2$, $J \equiv (\#\text{ levels of Factor B}) = 3$, $K \equiv (\text{Group size}) = 4$

2^{nd}, compute dof's: $\nu_A := I - 1 = 1$, $\nu_B := J - 1 = 2$, $\nu_{AB} := (I-1)(J-1) = 2$, $\nu_{res} := IJ(K-1) = 18$

3^{rd}: $MS_A := \frac{SS_A}{\nu_A} \approx 180.40167$, $MS_B := \frac{SS_B}{\nu_B} \approx 65.16116$, $MS_{AB} := \frac{SS_{AB}}{\nu_{AB}} \approx 31.79346$, $MS_{res} := \frac{SS_{res}}{\nu_{res}} \approx 1.5595$

4^{th}: $f_A := \frac{MS_A}{MS_{res}} = \frac{180.40167}{1.5595} \approx 115.68$, $f_B := \frac{MS_B}{MS_{res}} = \frac{65.16116}{1.5595} \approx 41.78$, $f_{AB} := \frac{MS_{AB}}{MS_{res}} = \frac{31.79346}{1.5595} \approx 20.39$

5^{th}: $f^*_{\nu_A,\nu_{res};\alpha} = f^*_{1,18;0.05} \overset{LOOKUP}{\approx} 4.414$, $f^*_{\nu_B,\nu_{res};\alpha} = f^*_{2,18;0.05} \overset{LOOKUP}{\approx} 3.555$, $f^*_{\nu_{AB},\nu_{res};\alpha} \overset{LOOKUP}{\approx} 3.555$

6^{th}, render appropriate decisions:

Since $f_A \approx 115.68 > 4.414 \approx f^*_{\nu_A,\nu_{res};\alpha}$, reject H_0^A, meaning bulb brand has a significant effect on lifetime.
Since $f_B \approx 41.78 > 3.555 \approx f^*_{\nu_B,\nu_{res};\alpha}$, reject H_0^B, meaning bulb wattage has a significant effect on lifetime.
Since $f_{AB} \approx 20.39 > 3.555 \approx f^*_{\nu_{AB},\nu_{res};\alpha}$, reject H_0^{AB}, meaning there is a significant interaction.

(d) Compute & interpret all the eta-squared & partial eta-squared effect size measures.

$\hat{\eta}_A^2 := \frac{SS_A}{SS_A + SS_B + SS_{AB} + SS_{res}} = \frac{180.40167}{402.38170} \approx \boxed{0.448} \implies$ 44.8% of the variation in bulb lifetime is due to brand.

$\hat{\eta}_B^2 := \frac{SS_B}{SS_A + SS_B + SS_{AB} + SS_{res}} = \frac{130.32231}{402.38170} \approx \boxed{0.324} \implies$ 32.4% of the variation in bulb lifetime is due to wattage.

$\hat{\eta}_{AB}^2 := \frac{SS_{AB}}{SS_A + SS_B + SS_{AB} + SS_{res}} = \frac{63.58691}{402.38170} \approx \boxed{0.158} \implies$ 15.8% of the variation in bulb lifetime is due to interaction.

$\hat{\eta}_{(A)}^2 := \frac{SS_A}{SS_A + SS_{res}} = \frac{180.40167}{180.40167 + 28.07085} \approx \boxed{0.865} \implies$ 86.5% of the variation possibly due to brand is truly due to it.

$\hat{\eta}_{(B)}^2 := \frac{SS_B}{SS_B + SS_{res}} = \frac{130.32231}{130.32231 + 28.07085} \approx \boxed{0.823} \implies$ 82.3% of the variation possibly due to wattage is truly due to it.

$\hat{\eta}_{(AB)}^2 := \frac{SS_{AB}}{SS_{AB} + SS_{res}} = \frac{63.58691}{63.58691 + 28.07085} \approx \boxed{0.694} \implies$ 69.4% of the variation possibly due to interaction is truly due to it.