2-FACTOR BALANCED EXPERIMENTS, MAIN EFFECTS & INTERACTIONS [DEVORE 11.2]

WHY 2F ANOVA AND NOT TWO 1F ANOVA'S:

Suppose one wishes to analyze a designed experiment involving two factors.

It seems reasonable to conduct two independent 1-Factor ANOVA's
– one on the 1^{st} factor (**factor A**), the other on the 2^{nd} factor (**factor B**).

Unfortunately, this is a poor strategy for the following reasons♣♡:

1. 2F ANOVA tests for an **interaction effect** – two 1F ANOVA's cannot.
   - (Definition and details later in this outline.)
2. 2F ANOVA results in more powerful F-tests than two 1F ANOVA's.
   - i.e. 2F ANOVA better explains variability than two 1F ANOVA's.
3. 2F ANOVA is more cost efficient than two 1F ANOVA's.
   - 2F ANOVA requires half as many measurements as two 1F ANOVA's.
4. 3F ANOVA generalizes easily from 2F ANOVA, not from two 1F ANOVA's.

♣ R.G. Lomax, D.L. Hahs-Vaughn, *Statistical Concepts: A 2^{nd} Course*, 4^{th} Ed., 2012.

♡ J.P. Stevens, *Intermediate Statistics: A Modern Approach*, 3^{rd} Ed., 2007.

2-FACTOR BALANCED EXPERIMENTS:

A 2-factor experiment with equal group sizes of $K > 1$ is called **balanced**.

A $I \times J$ 2F experiment means Factor A has I levels & Factor B has J levels.

FACTOR B: → **FACTOR A:** ↓	Level 1 ($x_{\bullet 1}$)	Level 2 ($x_{\bullet 2}$)
Level 1 ($x_{1\bullet}$)	$x_{111},\ x_{112}$	$x_{121},\ x_{122}$
Level 2 ($x_{2\bullet}$)	$x_{211},\ x_{212}$	$x_{221},\ x_{222}$
Level 3 ($x_{3\bullet}$)	$x_{311},\ x_{312}$	$x_{321},\ x_{322}$

Prototype 3×2 balanced experiment with $K = 2$

$\overline{x}_{11\bullet}$	=	Mean of all measurements at Level 1 of Factor A & Level 1 of Factor B	=	$(x_{111} + x_{112})/2$
$\overline{x}_{32\bullet}$	=	Mean of all measurements at Level 3 of Factor A & Level 2 of Factor B	=	$(x_{321} + x_{322})/2$
$\overline{x}_{1\bullet\bullet}$	=	Mean of all measurements at Level 1 of Factor A	=	$(x_{111} + x_{112} + x_{121} + x_{122})/4$
$\overline{x}_{3\bullet\bullet}$	=	Mean of all measurements at Level 3 of Factor A	=	$(x_{311} + x_{312} + x_{321} + x_{322})/4$
$\overline{x}_{\bullet 1\bullet}$	=	Mean of all measurements at Level 1 of Factor B	=	$(x_{111} + x_{112} + x_{211} + x_{212} + x_{311} + x_{312})/6$
$\overline{x}_{\bullet 2\bullet}$	=	Mean of all measurements at Level 2 of Factor B	=	$(x_{121} + x_{122} + x_{221} + x_{222} + x_{321} + x_{322})/6$

MAIN EFFECTS: Given a 2-Factor balanced completely randomized experiment.

A **main effect** of one factor is present if its effect at a fixed level is the same for all levels of the other factor.

INTERACTION EFFECTS: Given a 2-Factor balanced completely randomized experiment.

An **interaction**♠ **(effect)** is present if one factor's effect at a fixed level is not the same for all levels of the other factor.

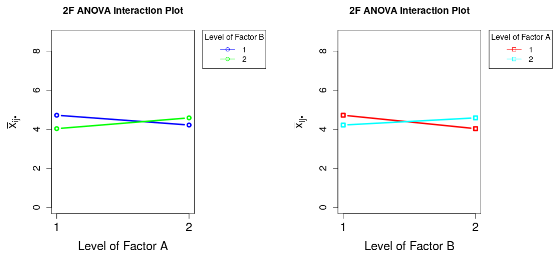
i.e. The combined levels of the two factors results in an effect in addition to any main effects of each factor alone.

i.e. A lack of interaction means the two factors' effects are independent.

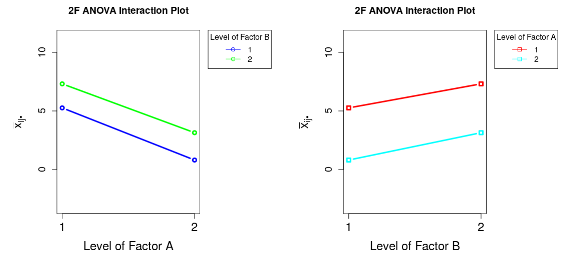
♠ J.P. Stevens, *Intermediate Statistics: A Modern Approach*, 3^{rd} Ed., 2007.

2x2 INTERACTION PLOTS (SEE SLIDES FOR MORE DETAILS) [DEVORE 11.2]

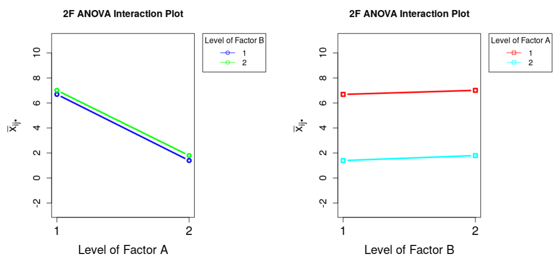
- **2x2 Plot (Given: A=no, B=no, AB=no)**



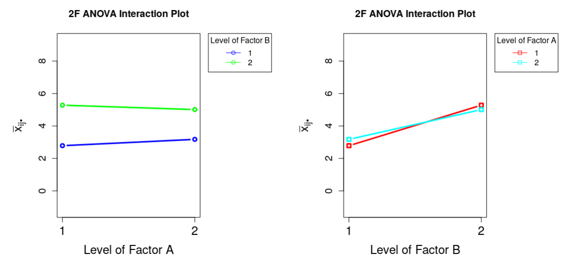
(Given: A=yes, B=yes, AB=no):



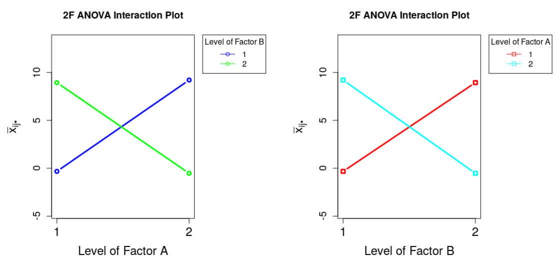
- **2x2 Plot (Given: A=yes, B=no, AB=no)**



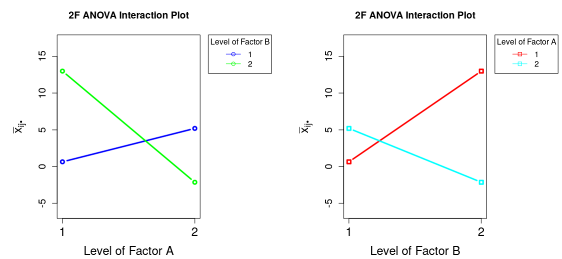
(Given: A=no, B=yes, AB=no):



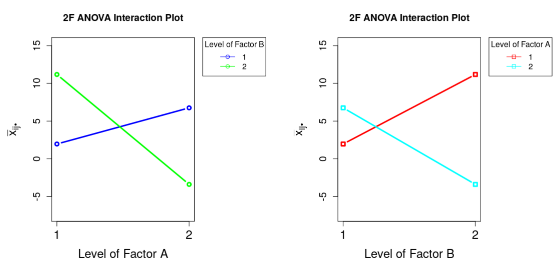
- **2x2 Plot (Given: A=no, B=no, AB=yes)**



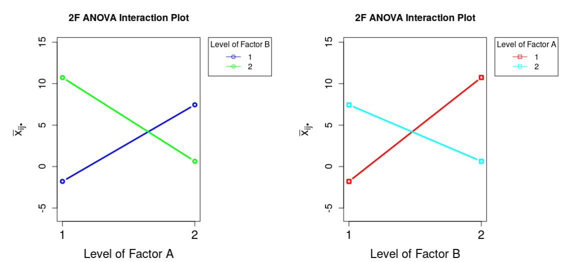
(Given: A=yes, B=yes, AB=yes):



- **2x2 Plot (Given: A=yes, B=no, AB=yes)**



(Given: A=no, B=yes, AB=yes):



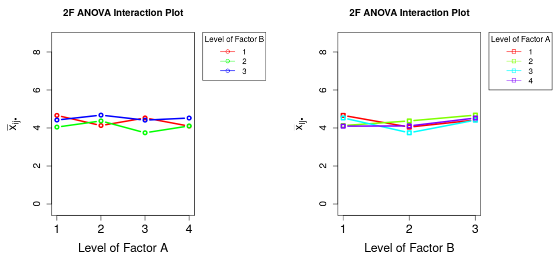
MORAL OF THE STORY REGARDING INTERACTION PLOTS:

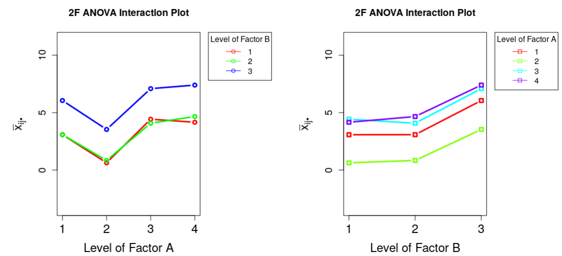
1. Use interaction plots to infer the presence of a significant interaction.
   - Widen plot's vertical axis limits by four times the estimated std deviation.
   - Otherwise, an interaction may appear when the vertical axis scale is small.
2. If there's no significant interaction present, the presence of main effects can be inferred.
3. If there is a significant interaction present, it's too hard to infer presence of main effects visually.
   - However, the actual 2F ANOVA may infer presence of main effects, but their proper interpretation is hard.
   - Moreover, 2F ANOVA can infer the presence of an interaction.

All this said, interaction plots are mainly used to determine the presence of a significant interaction before performing an ANOVA when the corresponding assumptions call for the presence or lack of said interaction.

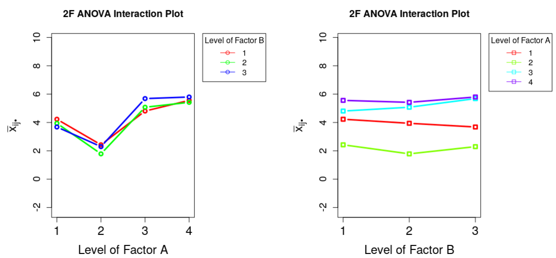
4x3 INTERACTION PLOTS (SEE SLIDES FOR MORE DETAILS) [DEVORE 11.2]

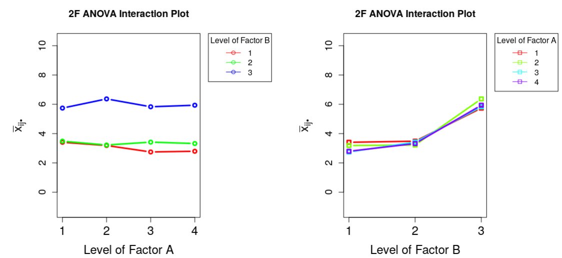
- **4x3 Plot (Given: A=no, B=no, AB=no)** **(Given: A=yes, B=yes, AB=no):**



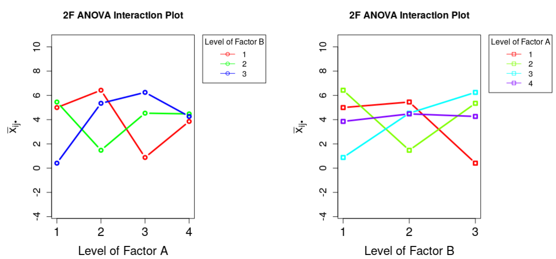


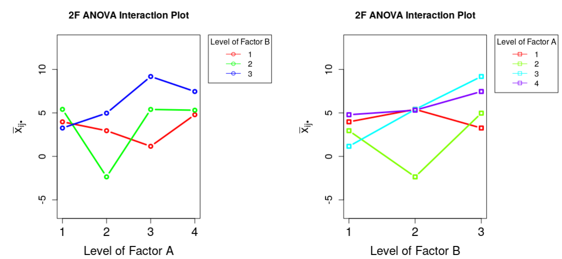
- **4x3 Plot (Given: A=yes, B=no, AB=no)** **(Given: A=no, B=yes, AB=no):**



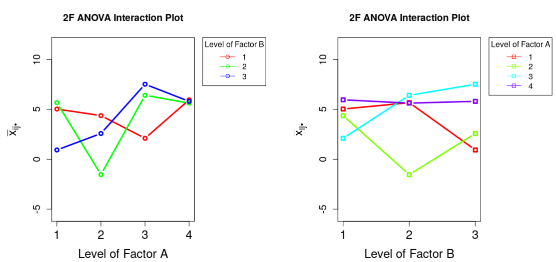


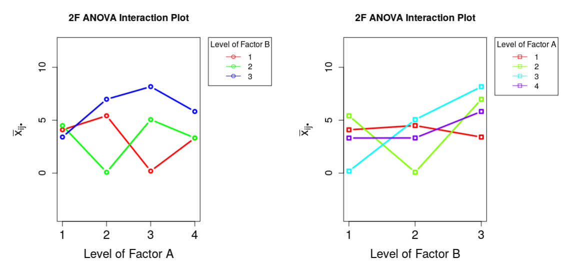
- **4x3 Plot (Given: A=no, B=no, AB=yes)** **(Given: A=yes, B=yes, AB=yes):**





- **4x3 Plot (Given: A=yes, B=no, AB=yes)** **(Given: A=no, B=yes, AB=yes):**





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2-FACTOR LINEAR MODELS & POINT ESTIMATORS [DEVORE 11.2]

2-FACTOR FIXED EFFECTS LINEAR MODEL (DEFINITION):

Given a 2-factor balanced experiment with IJ groups, each of size $K > 1$.

In particular, factor A has I levels and factor B has J levels.

Then, the **linear (statistical) model** for the experiment is defined as:

$$X_{ijk} = \mu + \alpha_i^A + \alpha_j^B + \gamma_{ij}^{AB} + E_{ijk} \quad \text{where} \quad E_{ijk} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$$

$$\begin{array}{ccl} X_{ijk} & \equiv & \text{rv for } k^{th} \text{ measurement at } (i,j)\text{-level of factors A \& B.} \\ \mu & \equiv & \text{Population grand mean of all } IJ \text{ population means} \\ (\alpha_i^A, \alpha_j^B) & \equiv & \text{Effect of } (i^{th}\text{-level factor A, } j^{th}\text{-level factor B)} \\ \gamma_{ij}^{AB} & \equiv & \text{Interaction between } (i,j)\text{-level factors A \& B} \\ E_{ijk} & \equiv & \text{Deviation of } X_{ijk} \text{ from } \mu \text{ due to random error} \end{array}$$

Fixed effects means all relevant levels of factor A are considered in model.

2-FACTOR LINEAR MODEL (LEAST-SQUARES ESTIMATORS – LSE's):

Given a 2-factor linear model: $X_{ijk} = \mu + \alpha_i^A + \alpha_j^B + \gamma_{ij}^{AB} + E_{ijk}$ where $E_{ijk} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$

Then:

(a) The **least-squares**♠♣ **estimators**†‡ **(LSE's)** for the model parameters are:

$$\begin{array}{ccc} \hat{\mu} & = & \overline{x}_{\bullet\bullet\bullet} \\ \hat{\alpha}_i^A & = & \overline{x}_{i\bullet\bullet} - \overline{x}_{\bullet\bullet\bullet} \\ \hat{\alpha}_j^B & = & \overline{x}_{\bullet j\bullet} - \overline{x}_{\bullet\bullet\bullet} \\ \hat{\gamma}_{ij}^{AB} & = & \overline{x}_{ij\bullet} - \overline{x}_{i\bullet\bullet} - \overline{x}_{\bullet j\bullet} + \overline{x}_{\bullet\bullet\bullet} \end{array} \quad \text{where} \quad \begin{array}{ccl} \overline{x}_{\bullet\bullet\bullet} & \equiv & \text{Grand sample mean} \\ \overline{x}_{i\bullet\bullet} & \equiv & \text{Mean of groups at } i^{th}\text{-lvl A} \\ \overline{x}_{\bullet j\bullet} & \equiv & \text{Mean of groups at } j^{th}\text{-lvl B} \\ \overline{x}_{ij\bullet} & \equiv & \text{Mean of } (i,j)\text{-lvl group} \end{array}$$

(b) For these least-squares estimators, it's required that $\sum_i \alpha_i^A = \sum_j \alpha_j^B = \sum_i \gamma_{ij}^{AB} = \sum_j \gamma_{ij}^{AB} = 0$.

(c) These least-squares estimators are all unbiased.

† A. Dean, D. Voss, D. Draguljić, *Design & Analysis of Experiments*, 2^{nd} Ed, Springer, 2017. (§3.4.3)

‡ D.C. Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.3.3, §3.10.1)

♠ A.M. Legendre, *Nouvelles Méthodes pour la Détermination des Orbites des Comètes*, 1806.

♣ Gauss, *Theoria Motus Corporum Coelestrium in Sectionibus Conicis Solem Ambientium*, 1809.

2-FACTOR LINEAR MODEL (PREDICTED RESPONSES & RESIDUALS):

Given a 2-factor linear model: $X_{ijk} = \mu + \alpha_i^A + \alpha_j^B + \gamma_{ij}^{AB} + E_{ijk}$ where $E_{ijk} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$

Then the corresponding **predicted responses**, denoted $\hat{x}_{ijk}$, are:

$$\hat{x}_{ijk} := \hat{\mu} + \hat{\alpha}_i^A + \hat{\alpha}_j^B + \hat{\gamma}_{ij}^{AB} = \overline{x}_{ij\bullet}$$

Moreover, the corresponding **residuals**, denoted x_{ijk}^{res}, are:

$$x_{ijk}^{res} := x_{ijk} - \hat{x}_{ijk} = x_{ijk} - \overline{x}_{ij\bullet}$$

2-FACTOR LINEAR MODEL (GAUSS[1]-MARKOV[2] THEOREM):

Given a 2-factor linear model: $X_{ijk} = \mu + \alpha_i^A + \alpha_j^B + \gamma_{ij}^{AB} + E_{ijk}$

Moreover, suppose the following conditions are all satisfied:

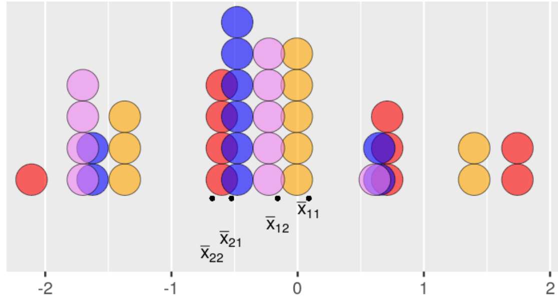
$$\begin{array}{cccl} \mathbb{E}[E_{ijk}] & = & 0 & \text{(errors are all centered at zero)} \\ \mathbb{V}[E_{ijk}] & = & \sigma^2 & \text{(errors all have the same finite variance)} \\ \mathbb{C}[E_{ijk}, E_{i'j'k'}] & = & 0 & \text{(errors are uncorrelated when } i \neq i' \text{ or } j \neq j' \text{ or } k \neq k'\text{)} \end{array}$$

Then, the least-squares estimators (LSE's) $\hat{\mu}, \hat{\alpha}_i^A, \hat{\alpha}_j^B, \hat{\gamma}_{ij}^{AB}$ are all BLUE's.

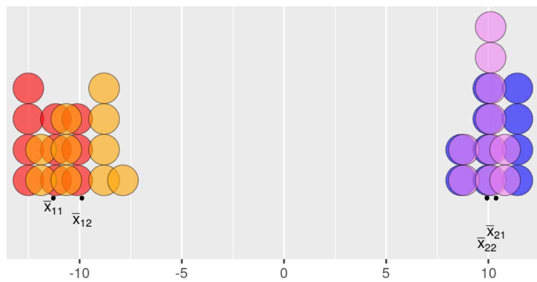
[1] C.F. Gauss, "Theoria Combinationis Observationum Erroribus Minimis Obnoxiae", (1823), 1-58.

[2] A.A. Markov, *Calculus of Probabilities*, 1^{st} Edition, 1900.

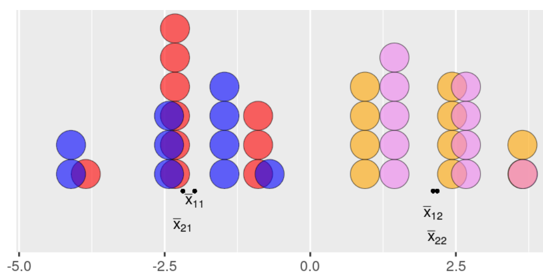
2-FACTOR ANOVA (MOTIVATION – NO INTERACTION) [DEVORE 11.2]



$$s_A^2/s_{within}^2 \ll 1 \implies \text{Factor A clearly has no significant main effect!}$$
$$s_B^2/s_{within}^2 \ll 1 \implies \text{Factor B clearly has no significant main effect!}$$
$$s_{AB}^2/s_{within}^2 \ll 1 \implies \text{Factors A \& B clearly have no interactive effect!}$$



$$s_A^2/s_{within}^2 \gg 1 \implies \text{Factor A clearly has a significant main effect!}$$
$$s_B^2/s_{within}^2 \ll 1 \implies \text{Factor B clearly has no significant main effect!}$$
$$s_{AB}^2/s_{within}^2 \ll 1 \implies \text{Factors A \& B clearly have no interactive effect!}$$



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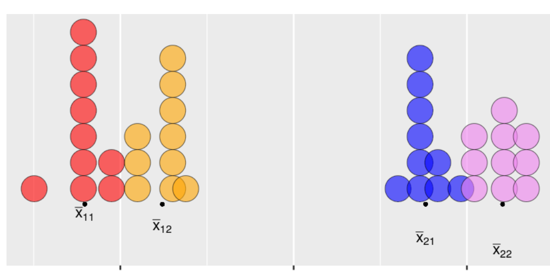

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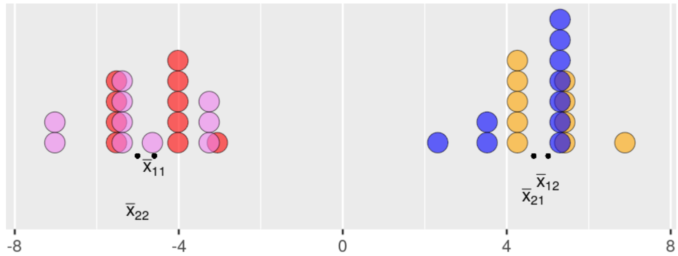
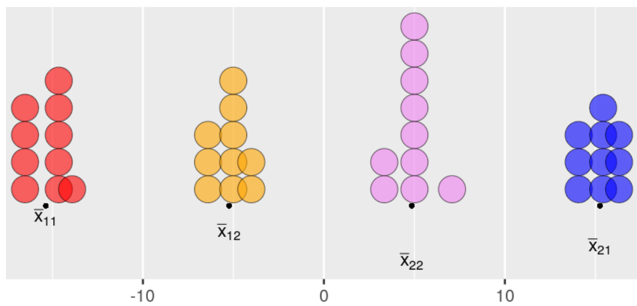
2-FACTOR ANOVA (MOTIVATION – WITH INTERACTION) [DEVORE 11.2]



$s_A^2/s_{within}^2 \ll 1 \implies$ Factor A clearly has no significant main effect!
$s_B^2/s_{within}^2 \ll 1 \implies$ Factor B clearly has no significant main effect!
$s_{AB}^2/s_{within}^2 \gg 1 \implies$ Factors A & B clearly have an interactive effect!



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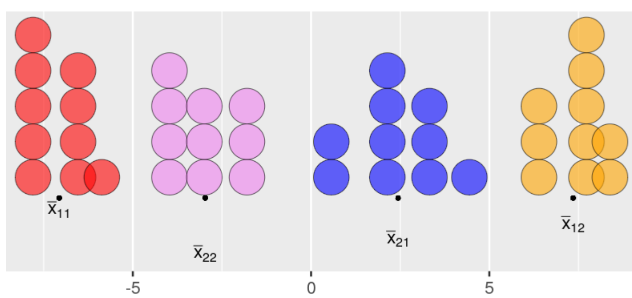

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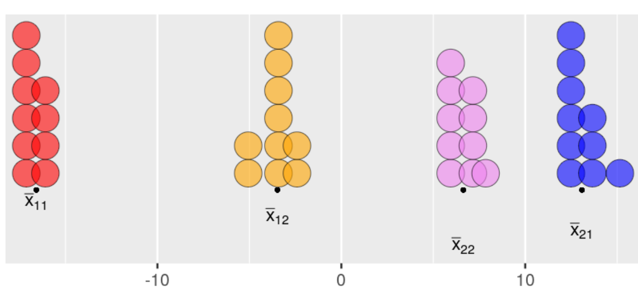

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2-FACTOR BALANCED COMPLETELY RANDOMIZED ANOVA (2F bcrANOVA) [DEVORE 11.2]

- **2F bcrANOVA (BALANCED COMPLETELY RANDOMIZED DESIGN):** As an example:
  - Collect 12 relevant experimental units (EU's): $\text{EU}_1, \text{EU}_2, \cdots, \text{EU}_{12}$
  - Produce a random shuffle sequence using software: $(6, 10;\ 3, 1;\ 5, 8;\ 11, 9;\ 7, 2;\ 12, 4)$
  - Use random shuffle sequence to assign the EU's into the IJ groups.
  - Measure each EU appropriately. (note the change in notation)

FACTOR B: → **FACTOR A:** ↓	Lvl 1	Lvl 2	Lvl 3
Level 1	EU_6, EU_{10}	EU_3, EU_1	EU_5, EU_8
Level 2	EU_{11}, EU_9	EU_7, EU_2	EU_{12}, EU_4

$\overset{MEASURE}{\Longrightarrow}$

B: → **A:** ↓	Lvl 1 $(x_{\bullet 1})$	Lvl 2 $(x_{\bullet 2})$	Lvl 3 $(x_{\bullet 3})$
Lvl 1 $(x_{1\bullet})$	x_{111}, x_{112}	x_{121}, x_{122}	x_{131}, x_{132}
Lvl 2 $(x_{2\bullet})$	x_{211}, x_{212}	x_{221}, x_{222}	x_{231}, x_{232}

- **2F bcrANOVA (FIXED EFFECTS MODEL ASSUMPTIONS):**
  - ⋆ (**2 Desired Factors**) Factor A has I levels & Factor B has J levels.
  - ⋆ (**All Factor Levels are Considered**) AKA Fixed Effects.
  - ⋆ (**Factors are Crossed**) IJ groups – one per (i, j)-level factor combination.
  - ⋆ (**Balanced Replication in Groups**) Each group has $K > 1$ units.
  - ⋆ (**Distinct Exp. Units**) All IJK units are distinct from each other.
  - ⋆ (**Random Assignment across Groups**)
  - ⋆ (**Independence**) All measurements on units are independent.
  - ⋆ (**Normality**) All groups are approximately normally distributed.
  - ⋆ (**Equal Variances**) All groups have approximately same variance.

  Mnemonic: **2DF AFLaC FaC BRiG DEU | RAaG | I.N.EV**

- **2F bcrANOVA (SUMS OF SQUARES "PARTITION" VARIATION):**

$$\underbrace{\text{SS}_{total}}_{\substack{Total\ Variation \\ in\ Experiment}} = \underbrace{\text{SS}_{A}}_{\substack{Variation\ due \\ to\ Factor\ A}} + \underbrace{\text{SS}_{B}}_{\substack{Variation\ due \\ to\ Factor\ B}} + \underbrace{\text{SS}_{AB}}_{\substack{Variation\ due \\ to\ Interaction}} + \underbrace{\text{SS}_{res}}_{\substack{Unexplained \\ Variation}}$$

$$\sum_{ijk}(x_{ijk} - \hat{\mu})^2 = \sum_{ijk}(\hat{\alpha}_i^A)^2 + \sum_{ijk}(\hat{\alpha}_j^B)^2 + \sum_{ijk}(\hat{\gamma}_{ij}^{AB})^2 + \sum_{ijk}(x_{ijk}^{res})^2$$

$$\underbrace{\nu}_{\substack{Total\ dof's\ in \\ Experiment}} = \underbrace{\nu_A}_{\substack{Factor\ A \\ dof's}} + \underbrace{\nu_B}_{\substack{Factor\ B \\ dof's}} + \underbrace{\nu_{AB}}_{\substack{Interaction \\ dof's}} + \underbrace{\nu_{res}}_{\substack{'Within\ Groups' \\ dof's}}$$

$$\nu = IJK - 1, \quad \nu_A = I - 1, \quad \nu_B = J - 1, \quad \nu_{AB} = (I-1)(J-1), \quad \nu_{res} = IJ(K-1)$$

- **2F bcrANOVA (EXPECTED MEAN SQUARES):**

$$\begin{array}{rrcl} (i) & \mathbb{E}[\text{MS}_{res}] & = & \sigma^2 \\ (ii) & \mathbb{E}[\text{MS}_{A}] & = & \sigma^2 + \frac{JK}{I-1}\sum_i(\alpha_i^A)^2 \\ (iii) & \mathbb{E}[\text{MS}_{B}] & = & \sigma^2 + \frac{IK}{J-1}\sum_j(\alpha_j^B)^2 \\ (iv) & \mathbb{E}[\text{MS}_{AB}] & = & \sigma^2 + \frac{K}{(I-1)(J-1)}\sum_i\sum_j(\gamma_{ij}^{AB})^2 \end{array}$$

- **2F bcrANOVA (POINT ESTIMATORS OF σ^2):**

$$\begin{array}{llll} (i) & \text{Regardless of the truthness of } H_0^A, H_0^B, H_0^{AB} & \Longrightarrow & \mathbb{E}[\text{MS}_{res}] = \sigma^2 \\ (ii) & H_0^A \text{ is true} \Longrightarrow \mathbb{E}[\text{MS}_A] = \sigma^2, & H_0^A \text{ is false} \Longrightarrow & \mathbb{E}[\text{MS}_A] > \sigma^2 \\ (iii) & H_0^B \text{ is true} \Longrightarrow \mathbb{E}[\text{MS}_B] = \sigma^2, & H_0^B \text{ is false} \Longrightarrow & \mathbb{E}[\text{MS}_B] > \sigma^2 \\ (iv) & H_0^{AB} \text{ is true} \Longrightarrow \mathbb{E}[\text{MS}_{AB}] = \sigma^2, & H_0^{AB} \text{ is false} \Longrightarrow & \mathbb{E}[\text{MS}_{AB}] > \sigma^2 \end{array}$$

2-FACTOR BALANCED COMPLETELY RANDOMIZED ANOVA (2F bcrANOVA) [DEVORE 11.2]

- **2F bcrANOVA (FIXED EFFECTS LINEAR MODEL):**

2F bcrANOVA Fixed Effects Linear Model		
(I, J)	$\equiv$	(# levels of factor A, # levels of factor B)
K	$\equiv$	# observations (replications) at each (i, j)-level of factors A & B
X_{ijk}	$\equiv$	rv for k^{th} observation at (i, j)-level of factors A & B
μ	$\equiv$	Mean average response over all levels of factors A & B
(α_i^A, α_j^B)	$\equiv$	(Effect of i^{th}-level factor A, Effect of j^{th}-level factor B)
γ_{ij}^{AB}	$\equiv$	Interaction between (i, j)-level factors A & B
E_{ijk}	$\equiv$	Deviation from μ due to random error

ASSUMPTIONS: $E_{ijk} \overset{iid}{\sim} \text{Normal}\left(0, \sigma^2\right)$

$$X_{ijk} = \mu + \alpha_i^A + \alpha_j^B + \gamma_{ij}^{AB} + E_{ijk} \quad \text{where} \quad \begin{cases} \sum_i \alpha_i^A = \sum_j \alpha_j^B = 0 \\ \sum_i \gamma_{ij}^{AB} = \sum_j \gamma_{ij}^{AB} = 0 \end{cases}$$

H_0^A :	All	$\alpha_i^A = 0$	H_0^B :	All	$\alpha_j^B = 0$	H_0^{AB} :	All	$\gamma_{ij}^{AB} = 0$
H_A^A :	Some	$\alpha_i^A \neq 0$	H_A^B :	Some	$\alpha_j^B \neq 0$	H_A^{AB} :	Some	$\gamma_{ij}^{AB} \neq 0$

- **2F bcrANOVA (*F*-TEST PROCEDURE):**

1. Determine df's: $\nu_A = I - 1,\ \nu_B = J - 1,\ \nu_{AB} = (I-1)(J-1),\ \nu_{res} = IJ(K-1)$
2. Compute Group Means (if not provided): $\overline{x}_{i\bullet\bullet} := \frac{1}{JK}\sum_j\sum_k x_{ijk}$, $\overline{x}_{\bullet j\bullet} := \frac{1}{IK}\sum_i\sum_k x_{ijk}$, $\overline{x}_{ij\bullet} := \frac{1}{K}\sum_k x_{ijk}$
3. Compute Grand Mean: $\overline{x}_{\bullet\bullet\bullet} := \frac{1}{IJK}\sum_i\sum_j\sum_k x_{ijk}$
4. Compute $\text{SS}_{res} := \sum_{ijk}(x_{ijk}^{res})^2 = \sum_i\sum_j\sum_k(x_{ijk} - \overline{x}_{ij\bullet})^2$
5. Compute $\text{SS}_A := \sum_{ijk}(\hat{\alpha}_i^A)^2 = \sum_i\sum_j\sum_k(\overline{x}_{i\bullet\bullet} - \overline{x}_{\bullet\bullet\bullet})^2$
6. Compute $\text{SS}_B := \sum_{ijk}(\hat{\alpha}_j^B)^2 = \sum_i\sum_j\sum_k(\overline{x}_{\bullet j\bullet} - \overline{x}_{\bullet\bullet\bullet})^2$
7. Compute $\text{SS}_{AB} := \sum_{ijk}(\hat{\gamma}_{ij}^{AB})^2 = \sum_i\sum_j\sum_k(\overline{x}_{ij\bullet} - \overline{x}_{i\bullet\bullet} - \overline{x}_{\bullet j\bullet} + \overline{x}_{\bullet\bullet\bullet})^2$

   (Optional) $\text{SS}_{total} := \sum_{ijk}(x_{ijk} - \hat{\mu})^2 = \sum_i\sum_j\sum_k(x_{ijk} - \overline{x}_{\bullet\bullet\bullet})^2$

8. Compute Mean Squares: $\text{MS}_{res} := \dfrac{\text{SS}_{res}}{\nu_{res}}$, $\text{MS}_A := \dfrac{\text{SS}_A}{\nu_A}$, $\text{MS}_B = \dfrac{\text{SS}_B}{\nu_B}$, $\text{MS}_{AB} = \dfrac{\text{SS}_{AB}}{\nu_{AB}}$
9. Compute Test Statistic Values: $f_A = \dfrac{\text{MS}_A}{\text{MS}_{res}}$, $f_B = \dfrac{\text{MS}_B}{\text{MS}_{res}}$, $f_{AB} = \dfrac{\text{MS}_{AB}}{\text{MS}_{res}}$
10. (if using software) Compute P-values: $\begin{cases} p_A := \mathbb{P}(F > f_A) \approx 1 - \Phi_F(f_A; \nu_A, \nu_{res}) \\ p_B := \mathbb{P}(F > f_B) \approx 1 - \Phi_F(f_B; \nu_B, \nu_{res}) \\ p_{AB} := \mathbb{P}(F > f_{AB}) \approx 1 - \Phi_F(f_{AB}; \nu_{AB}, \nu_{res}) \end{cases}$
11. Render Decisions: $\begin{cases} \text{If } p_A \leq \alpha \text{ or } f_A > f^*_{\nu_A, \nu_{res}; \alpha} \text{ then reject } H_0^A \text{ else accept } H_0^A. \\ \text{If } p_B \leq \alpha \text{ or } f_B > f^*_{\nu_B, \nu_{res}; \alpha} \text{ then reject } H_0^B \text{ else accept } H_0^B. \\ \text{If } p_{AB} \leq \alpha \text{ or } f_{AB} > f^*_{\nu_{AB}, \nu_{res}; \alpha} \text{ then reject } H_0^{AB} \text{ else accept } H_0^{AB}. \end{cases}$

- **2F bcrANOVA (SUMMARY TABLE):**

2F bcrANOVA Table (Significance Level α)

Variation Source	**df**	**Sum of Squares**	**Mean Square**	***F* Stat Value**	**P-value**	**Decision**
A	ν_A	SS_A	MS_A	f_A	p_A	Acc/Rej H_0^A
B	ν_B	SS_B	MS_B	f_B	p_B	Acc/Rej H_0^B
AB	ν_{AB}	SS_{AB}	MS_{AB}	f_{AB}	p_{AB}	Acc/Rej H_0^{AB}
Unknown	ν_{res}	SS_{res}	MS_{res}			
Total	ν	SS_{total}				

2F bcrANOVA (EFFECT SIZE MEASURES) [DEVORE 11.2]

- **2F bcrANOVA (EFFECT SIZE MEASURES & THEIR INTERPRETATIONS):**

YEAR	NAME	MEASURE	INTERPRETATION:
1925†	Fisher (Eta-Squared)	$\hat{\eta}^2_A := \frac{SS_A}{SS_{total}} = 0.38$	38% of the variation in the reponse is due to Factor A
		$\hat{\eta}^2_B := \frac{SS_B}{SS_{total}} = 0.02$	2% of the variation in the reponse is due to Factor B
		$\hat{\eta}^2_{AB} := \frac{SS_{AB}}{SS_{total}} = 0.27$	27% of the variation in the reponse is due to Interaction AB
		$\hat{\eta}^2_{res} := \frac{SS_{res}}{SS_{total}} = 0.33$	33% of the variation in the reponse is unexplained with experiment
1965‡	Cohen♠♣ (Partial η^2)	$\hat{\eta}^2_{(A)} := \frac{SS_A}{SS_A+SS_{res}} = 0.43$	43% of the variation possibly due to A is actually due to A
		$\hat{\eta}^2_{(B)} := \frac{SS_B}{SS_B+SS_{res}} = 0.65$	65% of the variation possibly due to B is actually due to B
		$\hat{\eta}^2_{(AB)} := \frac{SS_{AB}}{SS_{AB}+SS_{res}} = 0.31$	31% of the variation possibly due to AB is actually due to AB

$$\hat{\eta}^2_A + \hat{\eta}^2_B + \hat{\eta}^2_{AB} + \hat{\eta}^2_{res} = 1 \quad \text{but} \quad \hat{\eta}^2_{(A)} + \hat{\eta}^2_{(B)} + \hat{\eta}^2_{(AB)} > 1$$

†R.A. Fisher, *Statistical Methods for Reasearch Workers*, 1925.

‡B.B. Wolman (Ed.), *Handbook of Clinical Psychology*, 1965. (§5 by J. Cohen)

♠F.J. Gravetter, L.B. Wallnau, *Statistics for the Behavioral Sciences*, 7^{th} Ed., 2007.

♣R.G. Lomax, D.L. Hahs-Vaughn, *Statistical Concepts: A 2^{nd} Course*, 4^{th} Ed., 2012.

- **2F bcrANOVA (MORE EFFECT SIZE MEASURES):**

YEAR	NAME	MEASURE
1963◇	Hays (Omega-Squared)	$\hat{\omega}^2_A := \frac{SS_A - \nu_A MS_{res}}{SS_{total}+MS_{res}}$
		$\hat{\omega}^2_B := \frac{SS_B - \nu_B MS_{res}}{SS_{total}+MS_{res}}$
		$\hat{\omega}^2_{AB} := \frac{SS_{AB} - \nu_{AB} MS_{res}}{SS_{total}+MS_{res}}$
1979♡	Keren-Lewis (Partial ω^2)	$\hat{\omega}^2_{(A)} := \frac{SS_A - \nu_A MS_{res}}{SS_A+(n-\nu_A)MS_{res}}$
		$\hat{\omega}^2_{(B)} := \frac{SS_B - \nu_B MS_{res}}{SS_B+(n-\nu_B)MS_{res}}$
		$\hat{\omega}^2_{(AB)} := \frac{SS_{AB} - \nu_{AB} MS_{res}}{SS_{AB}+(n-\nu_{AB})MS_{res}}$

$$n := IJK = (1+\nu_A)(1+\nu_B)K$$

◇W.L. Hays, *Statistics for Psychologists*, 1963.

♡G. Keren, C. Lewis, "Partial Omega Squared for ANOVA Designs", *Educational & Psych. Measurement*, **39** (1979), 119-128.

ETA-SQUARED OR PARTIAL ETA-SQUARED?

There has been discussion regarding which effect size measure (eta-squared & partial eta-squared) is better for multi-factor ANOVA – the short answer being it depends on the particular multi-factor design(s) and whether meta-analyses will be performed[1,2,3,4].

To play it safe, we shall always report <u>both</u> η^2 & $\eta^2_{(\cdot)}$. Ditto for ω^2 & $\omega^2_{(\cdot)}$.

[1] J. Cohen, "Eta-Squared and Partial Eta-Squared in Fixed Factor ANOVA Designs", *Edu. & Psy. Meas.*, **33** (1973), 107-112.

[2] T.R. Levine, C.R. Hullett, "Eta Squared, Partial Eta Squared, and Misreporting of Effect Size in Communication Research", *Human Communication Research*, **28** (2002), 612-625.

[3] S. Olejnik, J. Algina, "Generalized Eta and Omega Squared Statistics: Measures of Effect Size for Some Common Research Designs", *Psychological Methods*, **8** (2003), 434-447.

[4] C.A. Pierce, R.A. Block, H. Aguinis, "Cautionary Note on Reporting Eta-Squared Values from Multifactor ANOVA Designs", *Educational and Psychological Measurement*, **64** (2004), 916-924.

2F bcrANOVA (POST-HOC COMPARISONS) [DEVORE 11.2]

- **2F bcrANOVA (TUKEY POST-HOC COMPARISONS FOR FACTOR A'S MAIN EFFECT – NO INTERACTION):**

  Given a 2-factor experiment with I levels of factor A, J levels of factor B, and each group has $K > 1$ measurements.
  Moreover, 2F bcrANOVA accepts H_0^{AB} and rejects H_0^A at significance level α. $[\nu_{res} := IJ(K-1)]$
  Then, to determine which levels of factor A significantly differ:

  1. Compute the factor A significant difference width:

  $$w_A = q^*_{I,\nu_{res};\alpha} \cdot \sqrt{\text{MS}_{res}/(JK)}$$

  2. Sort the I factor A level means in ascending order:

  $$\overline{x}_{(1)\bullet\bullet} \leq \overline{x}_{(2)\bullet\bullet} \leq \cdots \leq \overline{x}_{(I)\bullet\bullet}$$

  3. For each sorted factor A level mean $\overline{x}_{(i)\bullet\bullet}$:
     - If $\overline{x}_{(i+1)\bullet\bullet} \notin \left[\overline{x}_{(i)\bullet\bullet},\ \overline{x}_{(i)\bullet\bullet} + w_A\right]$, repeat STEP 3 with next sorted mean.
     - Else, underline $\overline{x}_{(i)\bullet\bullet}$ and all larger means within a distance of w_A with new line.

- **2F bcrANOVA (TUKEY POST-HOC COMPARISONS FOR FACTOR B'S MAIN EFFECT – NO INTERACTION):**

  Given a 2-factor experiment with I levels of factor A, J levels of factor B, and each group has $K > 1$ measurements.
  Moreover, 2F bcrANOVA accepts H_0^{AB} and rejects H_0^B at significance level α. $[\nu_{res} := IJ(K-1)]$
  Then, to determine which levels of factor B significantly differ:

  1. Compute the factor B significant difference width:

  $$w_B = q^*_{J,\nu_{res};\alpha} \cdot \sqrt{\text{MS}_{res}/(IK)}$$

  2. Sort the J factor B level means in ascending order:

  $$\overline{x}_{\bullet(1)\bullet} \leq \overline{x}_{\bullet(2)\bullet} \leq \cdots \leq \overline{x}_{\bullet(J)\bullet}$$

  3. For each sorted factor B level mean $\overline{x}_{\bullet(j)\bullet}$:
     - If $\overline{x}_{\bullet(j+1)\bullet} \notin \left[\overline{x}_{\bullet(j)\bullet},\ \overline{x}_{\bullet(j)\bullet} + w_B\right]$, repeat STEP 3 with next sorted mean.
     - Else, underline $\overline{x}_{\bullet(j)\bullet}$ and all larger means within a distance of w_B with new line.

2F bcrANOVA (POST-HOC COMPARISONS WHEN THERE'S A SIGNIFICANT INTERACTION):

Post-hoc comparisons when there is a statistically significant interaction (i.e. 2F bcrANOVA rejects H_0^{AB}) are far trickier and, hence, beyond the scope of this course.

Interested readers may consult any of the following:

L.E. Toothaker, *Multiple Comparison Procedures*, SAGE, 1992. (Ch 5)

P.H. Westfall *et al*, *Multiple Comparisons & Multiple Tests using SAS*, SAS Institute, 1999. (§9.2.4)

Y. Hochberg *et al*, *Multiple Comparison Procedures*, Wiley, 1987. (§10.5)

G. Keppel, *Design and Analysis: A Researcher's Handbook*, Pearson, 1991.

R.J. Boik, "The Analysis of 2-Factor Interactions in Fixed Effects Linear Models", *Journal of Educational Stats.*, **18** (1993), 1-40.

EX 11.2.1: The lifetimes of 24 light bulbs, available in two brands and three wattages, were randomized and then measured:

BULB LIFETIME (in years)				
WATTAGE: → **BRAND:** ↓	60-Watt ($x_{\bullet 1}$)	75-Watt ($x_{\bullet 2}$)	100-Watt ($x_{\bullet 3}$)	**TOTAL** ($\sum_j \sum_k x_{ijk}$)
Brand 1 ($x_{1\bullet}$)	9.23, 7.64, 8.59, 7.66	8.54, 5.98, 8.15, 8.30	1.29, 3.13, 1.42, 3.28	**73.21**
Brand 2 ($x_{2\bullet}$)	14.54, 13.77, 15.43, 14.20	10.82, 10.84, 12.86, 13.81	9.65, 9.00, 8.24, 8.61	**141.77**
TOTAL ($\sum_i \sum_k x_{ijk}$)	**91.06**	**79.30**	**44.62**	$\sum_i \sum_j \sum_k x_{ijk} =$ **214.98**

(a) Formulate this experiment as a 2-factor fixed effects linear model. In this context, what does "fixed effects" assume?

(b) State the appropriate null & alternative hypotheses for factor A, factor B and interaction AB.

(c) Perform a 2-Factor Balanced Completely Randomized ANOVA (2F bcrANOVA) with ($\alpha = 0.01$) significance level.

To save time and tedium: $\text{SS}_{total} = 366.15840$, $\text{SS}_{res} = 18.75875$, (and utilize the row & column totals in the table!)

(d) Perform & interpret the appropriate Tukey Complete Pairwise Post-Hoc Comparison.

EX 11.2.2: The lifetimes of 24 light bulbs, available in two brands and three wattages, were randomized and then measured:

BULB LIFETIME (in years)				
WATTAGE: → **BRAND:** ↓	60-Watt ($x_{\bullet 1}$)	75-Watt ($x_{\bullet 2}$)	100-Watt ($x_{\bullet 3}$)	**TOTAL** ($\sum_j \sum_k x_{ijk}$)
Brand 1 ($x_{1\bullet}$)	10.78, 9.87, 12.37, 8.38	5.79, 4.35, 7.02, 5.16	2.51, 2.70, 5.05, 2.46	**76.44**
Brand 2 ($x_{2\bullet}$)	11.31, 12.63, 11.60, 12.15	16.31, 14.33, 14.66, 15.19	7.73, 8.27, 7.20, 10.86	**142.24**
TOTAL ($\sum_i \sum_k x_{ijk}$)	**89.09**	**82.81**	**46.78**	$\sum_i \sum_j \sum_k x_{ijk} =$ **218.68**

(a) Formulate this experiment as a 2-factor fixed effects linear model.

(b) State the appropriate null & alternative hypotheses for factor A, factor B and interaction AB.

(c) Perform a 2-Factor Balanced Completely Randomized ANOVA (2F bcrANOVA) with ($\alpha = 0.05$) significance level.

To save time: $\text{SS}_{total} = 402.38170$, $\text{SS}_A \approx 180.40167$, $\text{SS}_B \approx 130.32231$, $\text{SS}_{AB} \approx 63.58691$, $\text{SS}_{res} = 28.07085$

(d) Compute & interpret all the eta-squared & partial eta-squared effect size measures.