MOTIVATION: GAMMA FUNCTION

GAMMA FUNCTION INTEGRAL DEFINITION (GFID):

There are many definitions of the **gamma function**, but we will only consider Euler's integral definition devised 1730:

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \quad \text{where} \ \ \alpha > 0$$

GAMMA FUNCTION CONVERGENCE THEOREM (GFCT):

$$\text{The improper integral} \quad I := \int_0^\infty x^{\alpha-1} e^{-x}\, dx \quad \text{converges for} \ \ \alpha > 0$$

GAMMA FUNCTION PROPERTIES:

$$\begin{aligned}
\Gamma(1) &= 1 \\
\Gamma(\alpha) &= (\alpha-1)\cdot\Gamma(\alpha-1) \quad \text{where} \quad \alpha > 1 \\
\Gamma(n) &= (n-1)! \quad \text{where} \quad n \in \mathbb{Z}_+ \\
\\
\Gamma\left(\tfrac{1}{2}\right) &= \sqrt{\pi} \\
\Gamma\left(n+\tfrac{1}{2}\right) &= \left(n-\tfrac{1}{2}\right)\cdot\Gamma\left(n-\tfrac{1}{2}\right) \quad \text{where} \quad n \in \mathbb{Z}_+
\end{aligned}$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}\cdot[(1)(3)(5)\cdots(2n-5)(2n-3)(2n-1)] \quad \text{where} \quad n \in \mathbb{Z}_+$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\cdot\frac{\Gamma(2n)}{\Gamma(n)} \quad \text{where} \quad n \in \mathbb{Z}_+$$

E. Artin, *The Gamma Function / Einführung in die Theorie der Gammafunktion*, 1964/1931. (Ch2)

H.B. Dwight, *Tables and Integrals and Other Math. Data*, 4^{th} Ed, 1961. (Ch12, entry 850.8)

B.O. Peirce, R.M. Foster, *A Short Table of Integrals*, 4^{th} Ed, 1956. (Part III, entry 791)

L.A. Pipes, *Applied Math. for Engineers & Physicists*, McGraw-Hill, 1946. (Ch XII, §4)

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GAMMA FUNCTION CONVERGENCE THEOREM (GFCT):

$$\text{The improper integral} \quad I := \int_0^\infty x^{\alpha-1}e^{-x}\,dx \quad \text{converges for} \quad \alpha > 0$$

PROOF: Fix $\alpha > 0$ and split integral as $I = \int_0^1 x^{\alpha-1}e^{-x}\,dx + \int_1^\infty x^{\alpha-1}e^{-x}\,dx \equiv I_0 + I_\infty$

Consider integral $I_\epsilon := \int_\epsilon^1 x^{\alpha-1}e^{-x}\,dx$ where $0 < \epsilon \ll 1$. Then, $e^{-x} < 1\ \forall x \in (0,1]$

$$\implies I_\epsilon := \int_\epsilon^1 x^{\alpha-1}e^{-x}\,dx < \int_\epsilon^1 x^{\alpha-1}\,dx = \left[\frac{x^\alpha}{\alpha}\right]_{x=\epsilon}^{x=1} \overset{FTC}{=} \frac{1}{\alpha} - \frac{\epsilon^\alpha}{\alpha} < \frac{1}{\alpha} < \infty \implies I_\epsilon \text{ is bounded}$$

Since, for fixed $\alpha > 0$, the integrand of I_ϵ, $x^{\alpha-1}e^{-x}$ is always positive for all $x \in (0,1]$, decreasing the value of ϵ towards zero (with fixed $\alpha > 0$) results in I_ϵ monotonically increasing.

Therefore, since integral I_ϵ is bounded above and monotically increases as $\epsilon \downarrow 0$,

$$\lim_{\epsilon\downarrow 0} I_\epsilon = I_0 \equiv \int_0^1 x^{\alpha-1}e^{-x}\,dx \quad \text{converges for all } \alpha > 0.$$

Consider integral $I_\omega := \int_1^\omega x^{\alpha-1}e^{-x}\,dx$ where $1 \ll \omega < \infty$. Recall: $e^x \overset{TS}{=} \sum_{k=0}^\infty \frac{x^k}{k!}\quad \forall x \in \mathbb{R}$

$$\implies e^x > \frac{x^K}{K!} \quad \forall K \geq 0 \implies \text{ in particular, } e^x > \frac{x^{\alpha+1}}{(\alpha+1)!} \implies e^{-x} = \frac{1}{e^x} < \frac{(\alpha+1)!}{x^{\alpha+1}} \quad \forall x \in [1,\infty)$$

$$\implies x^{\alpha-1}e^{-x} < \frac{(\alpha+1)!}{x^{\alpha+1+(1-\alpha)}} = \frac{(\alpha+1)!}{x^2} \quad \forall x \in [1,\infty)$$

$$\implies I_\omega < \int_1^\omega \frac{(\alpha+1)!}{x^2}\,dx = \left[-\frac{(\alpha+1)!}{x}\right]_{x=1}^{x=\omega} \overset{FTC}{=} (\alpha+1)! \cdot \left(1 - \frac{1}{\omega}\right) < (\alpha+1)! \implies I_\omega \text{ is bounded}$$

Since, for fixed $\alpha > 0$, the integrand of I_ω, $x^{\alpha-1}e^{-x}$ is always positive for all $x \in [1,\infty)$, increasing the value of ω towards infinity (with fixed $\alpha > 0$) results in I_ω monotonically increasing.

Therefore, since integral I_ω is bounded above and monotically increases as $\omega \uparrow \infty$,

$$\lim_{\omega\uparrow\infty} I_\omega = I_\infty \equiv \int_1^\infty x^{\alpha-1}e^{-x}\,dx \quad \text{converges for all } \alpha > 0.$$

Finally, since integrals I_0 and I_∞ have each been shown to converge for all $\alpha > 0$, their sum $I := \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ also converges for all $\alpha > 0$. □

E. Artin, *The Gamma Function / Einführung in die Theorie der Gammafunktion*, 1964/1931. (Ch2)

MOTIVATION: GAMMA FUNCTION

GAMMA FUNCTION OF ONE (GF1): $\Gamma(1) = 1$

PROOF: $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \Big[-e^{-x}\Big]_{x=0}^{x\to\infty} \overset{FTC}{=} -\lim_{x\to\infty} e^{-x} - (-e^0) = -0 - (-1) = 1 \quad \Box$

GAMMA FUNCTION REDUCTION FORMULA (GFRF): $\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1) \quad \text{where } \alpha > 1$

PROOF: Apply Integration by Parts.

(IBP): Let $\begin{cases} u = x^{\alpha-1} \\ dv = e^{-x}\, dx \end{cases} \iff \begin{cases} du = (\alpha-1)x^{\alpha-2}\, dx \\ v = -e^{-x} \end{cases}$

$$\begin{aligned}
\implies \Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}\, dx \equiv \int_0^\infty u\, dv \overset{IBP}{=} \Big[u(x)v(x)\Big]_{x=0}^{x\to\infty} - \int_0^\infty v\, du \\
&= \Big[-x^{\alpha-1}e^{-x}\Big]_{x=0}^{x\to\infty} - \int_0^\infty \left(-e^{-x}\right)\cdot(\alpha-1)x^{\alpha-2}\, dx \\
&\overset{FTC}{=} \left[-\lim_{x\to\infty}\left(\frac{x^{\alpha-1}}{e^x}\right) - (-0)\right] + (\alpha-1)\cdot\int_0^\infty x^{\alpha-2}e^{-x}\, dx \\
&\overset{(*)}{=} 0 + (\alpha-1)\cdot\int_0^\infty x^{(\alpha-1)-1}e^{-x}\, dx \qquad (*) \text{ L'Hospital's Rule or Tower of Power} \\
&\overset{\Gamma}{=} (\alpha-1)\cdot\Gamma(\alpha-1) \qquad \Box
\end{aligned}$$

GAMMA FUNCTION OF HALF OF UNITY (GF$\frac{1}{2}$): $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

PROOF: Apply two appropriate Changes of Variables leading to Standard Normal pdf.

(CV1): Let $x = y^2 \iff dx = 2y\, dy \implies \begin{cases} x = 0 \iff y = \sqrt{0} = 0 \\ x = \infty \iff y = \sqrt{\infty} = \infty \end{cases}$

$$\implies \Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\, dx \overset{CV1}{=} \int_0^\infty y^{2(\alpha-1)}e^{-y^2}\cdot(2y\, dy) = 2\cdot\int_0^\infty y^{2\alpha-1}e^{-y^2}\, dy$$

(CV2): Let $y = \frac{z}{\sqrt{2}} \iff y^2 = \frac{z^2}{2} \iff dy = \frac{1}{\sqrt{2}}\, dz \implies \begin{cases} y = 0 \iff z = 0\cdot\sqrt{2} = 0 \\ y = \infty \iff z = \infty\cdot\sqrt{2} = \infty \end{cases}$

$$\Gamma\left(\frac{1}{2}\right) \overset{CV1}{=} 2\cdot\int_0^\infty e^{-y^2}\, dy \overset{CV2}{=} 2\cdot\int_0^\infty \frac{1}{\sqrt{2}}e^{-z^2/2}\, dz \overset{CIO}{=} 2\sqrt{\pi}\cdot\int_0^\infty \underbrace{\frac{1}{\sqrt{2\pi}}e^{-z^2/2}}_{N(0,1)\ \text{pdf}}\, dz \overset{N(0,1)}{=} 2\sqrt{\pi}\cdot\frac{1}{2} = \sqrt{\pi}$$

$\therefore\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \Box$

L.A. Pipes, *Applied Mathematics for Engineers & Physicists*, McGraw-Hill, 1946. (Ch XII, §4)