Tukey Complete Pairwise Post-Hoc Comparison

Engineering Statistics II
Section 10.2

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PART I:

Gosset's Q Distribution

Gosset's Q Distribution

Definition

Notation	$Q \sim Q_{M,\nu}$
Parameters	$M \equiv$ # Groups $\nu \equiv$ # degrees of freedom
Support	$\text{Supp}(Q) = [0, \infty)$
pdf	$f_Q(q; M, \nu) := \frac{\sqrt{2\pi} \cdot M(M-1)\nu^{\nu/2}}{\Gamma(\nu/2) \cdot 2^{-1+\nu/2}} \cdot \int_0^{\infty} x^{\nu} \cdot \Phi'(\sqrt{\nu} \cdot x) \cdot \left[\int_{-\infty}^{\infty} \Phi'(u)\Phi'(u-qx)[\Phi(u) - \Phi(u-qx)]^{M-2} du\right] dx$
Model(s)	(Used exclusively for Statistical Inference)

$\Phi(\cdot) \equiv$ Std Normal cdf $\quad \Phi'(\cdot) \equiv$ Std Normal pdf $\quad \Gamma(\cdot) \equiv$ Gamma Function

Gosset's Q Distribution (Derivation)

Proposition

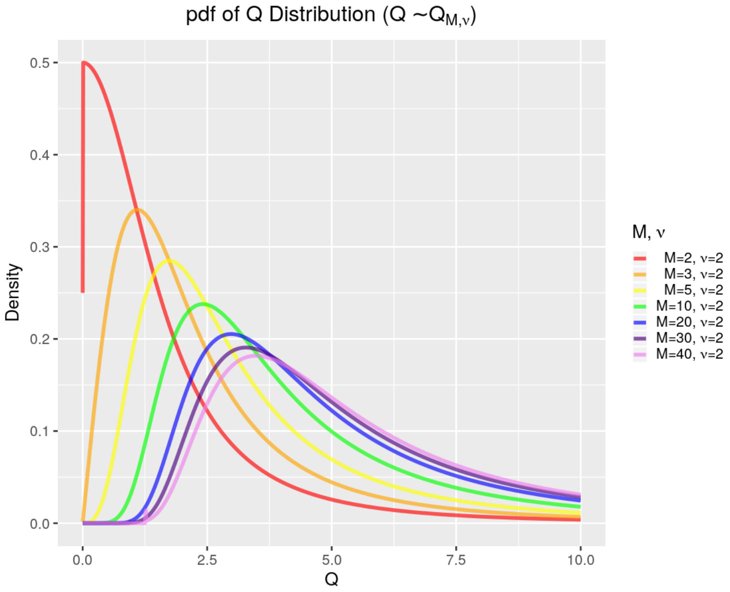
Given an experiment with M Normal(μ, σ^2) random samples each of size J. Define $\overline{X}_{(1)}, \overline{X}_{(M)}$ to be the smallest and largest sample means, respectively. Moreover, let S^2_{pool} be the pooled sample variance from the M samples.

*Then, a Q **distribution** (AKA **Studentized Range distribution**) can created as follows:*

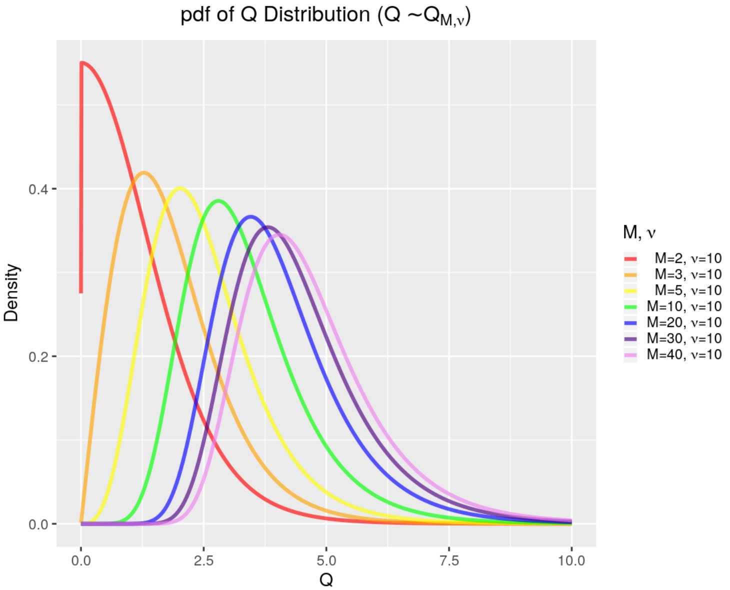
$$Q := \frac{\overline{X}_{(M)} - \overline{X}_{(1)}}{S_{pool}/\sqrt{J}} \sim Q_{M,M(J-1)}$$

PROOF: Beyond scope of course. Take **Mathematical Statistics**.

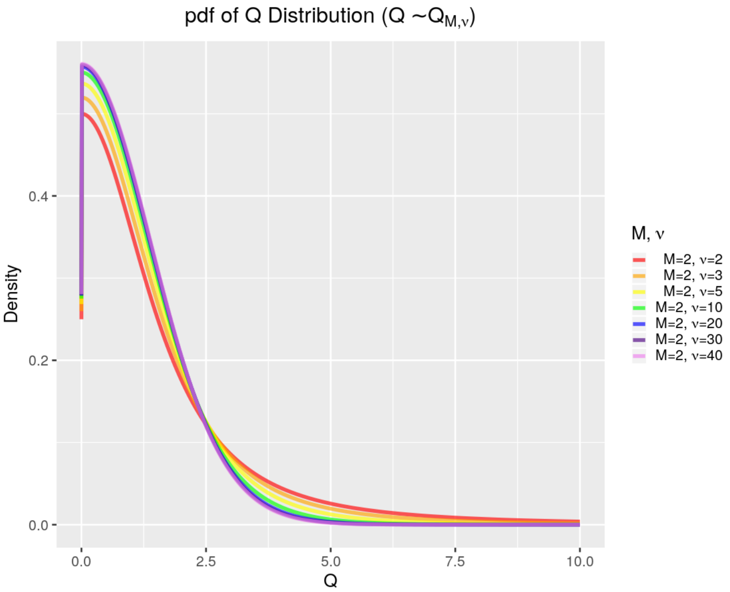
Plot of Q Distributions (M grows & $\nu = 2$ is fixed)



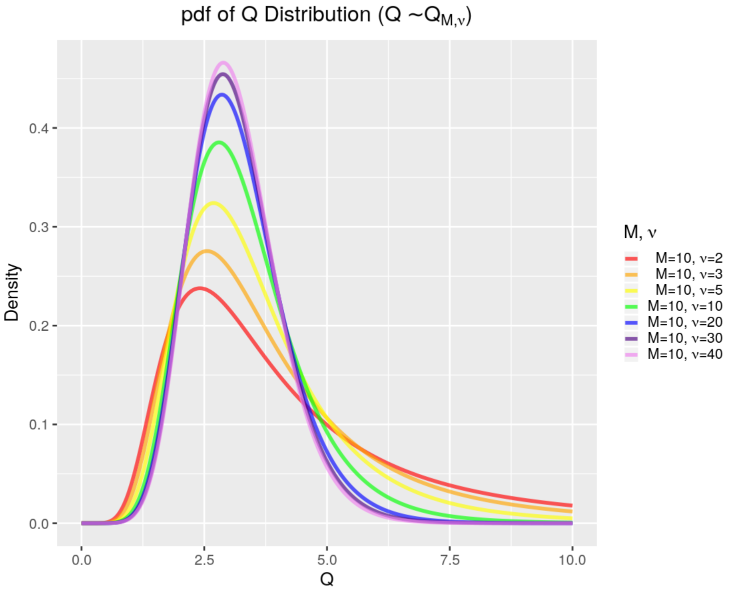
Plot of Q Distributions (M grows & $\nu = 10$ is fixed)



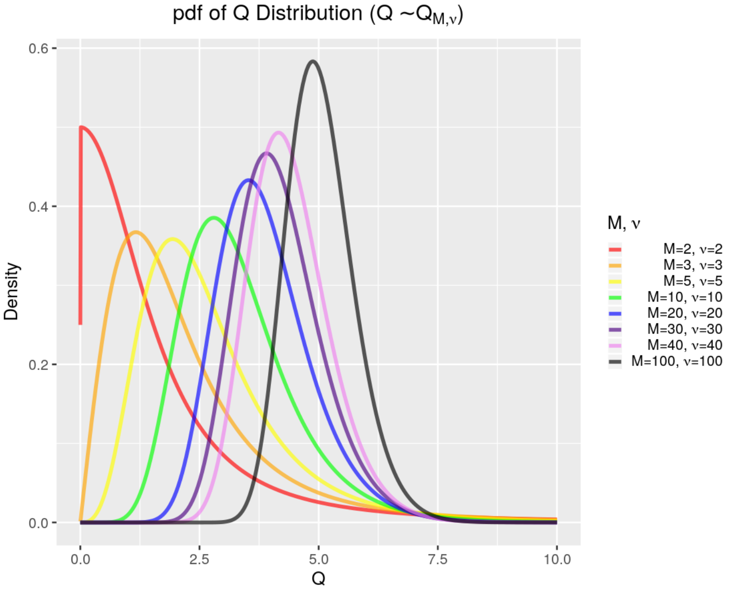
Plot of Q Distributions ($M = 2$ is fixed & ν grows)



Plot of Q Distributions ($M = 10$ is fixed & ν grows)



Plot of Q Distributions (M & ν both grow in unison)



Q-Cutoffs (AKA Q Critical Values) (Definition)

A key component to some ANOVA post-processing is the Q**-cutoff**:

Definition

$q^*_{M,\nu;\alpha}$ is called a Q**-cutoff** of the $Q_{M,\nu}$ distribution such that its upper-tail probability is exactly its subscript value α: (Here, $Q \sim Q_{M,\nu}$)

$$\mathbb{P}(Q > q^*_{M,\nu;\alpha}) = \alpha$$

NOTE: Do not confuse Q-cutoff $q^*_{M,\nu;\alpha}$ with Q percentile $q_{M,\nu;\alpha}$:

$$\mathbb{P}(Q \leq q_{M,\nu;\alpha}) = \alpha$$

Another name for Q-cutoff is Q **critical value**.

Q-Cutoffs Table $(\alpha = 0.1)$

$(\alpha = \mathbf{0.1})$ **GOSSET'S Q-CUTOFFS, $q^*_{M,\nu;\alpha}$** $\mathbb{P}(Q > q^*_{M,\nu;\alpha}) = \alpha$

ν \ M	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	3.328	4.463	5.200	5.740	6.164	6.512	6.807	7.062	7.286	7.486	7.667	7.831	7.981	8.119
4	3.015	3.971	4.588	5.037	5.390	5.680	5.926	6.139	6.327	6.494	6.645	6.782	6.908	7.024
5	2.850	3.713	4.265	4.666	4.981	5.239	5.459	5.649	5.816	5.965	6.100	6.223	6.335	6.439
6	2.748	3.554	4.066	4.436	4.728	4.967	5.169	5.344	5.499	5.637	5.761	5.875	5.979	6.074
7	2.679	3.448	3.931	4.281	4.556	4.781	4.972	5.137	5.283	5.413	5.530	5.637	5.735	5.825
8	2.630	3.371	3.835	4.170	4.432	4.647	4.829	4.987	5.126	5.250	5.362	5.464	5.557	5.644
9	2.592	3.313	3.762	4.085	4.338	4.546	4.721	4.873	5.007	5.126	5.234	5.332	5.422	5.505
10	2.563	3.268	3.704	4.019	4.264	4.466	4.636	4.784	4.913	5.029	5.134	5.229	5.316	5.397
11	2.540	3.232	3.659	3.966	4.205	4.402	4.568	4.711	4.838	4.951	5.053	5.145	5.230	5.309
12	2.521	3.202	3.621	3.922	4.157	4.349	4.511	4.652	4.776	4.886	4.986	5.076	5.159	5.236
13	2.504	3.178	3.590	3.885	4.116	4.305	4.464	4.602	4.724	4.832	4.930	5.018	5.100	5.175
14	2.491	3.157	3.563	3.855	4.082	4.267	4.424	4.560	4.679	4.786	4.882	4.969	5.049	5.123
15	2.479	3.139	3.540	3.828	4.052	4.235	4.390	4.524	4.641	4.746	4.841	4.927	5.006	5.079
16	2.469	3.123	3.520	3.805	4.026	4.207	4.360	4.492	4.608	4.712	4.805	4.890	4.968	5.040
17	2.460	3.109	3.503	3.785	4.004	4.183	4.334	4.464	4.579	4.681	4.773	4.857	4.934	5.005

Q-Cutoffs Table $(\alpha = 0.05)$

$(\alpha = \mathbf{0.05})$ **GOSSET'S Q-CUTOFFS,** $q^*_{M,\nu;\alpha}$ $\mathbb{P}(Q > q^*_{M,\nu;\alpha}) = \alpha$

ν \ M	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	4.501	5.904	6.827	7.505	8.039	8.480	8.853	9.177	9.462	9.716	9.945	10.15	10.34	10.52
4	3.926	5.033	5.758	6.290	6.709	7.055	7.348	7.602	7.826	8.027	8.208	8.372	8.523	8.663
5	3.635	4.596	5.219	5.675	6.035	6.331	6.583	6.802	6.995	7.167	7.323	7.465	7.595	7.715
6	3.460	4.334	4.896	5.307	5.630	5.897	6.123	6.320	6.493	6.648	6.789	6.916	7.034	7.142
7	3.344	4.161	4.682	5.061	5.361	5.607	5.816	5.998	6.158	6.302	6.431	6.549	6.658	6.758
8	3.261	4.037	4.529	4.887	5.168	5.400	5.597	5.768	5.918	6.053	6.175	6.286	6.388	6.482
9	3.199	3.945	4.415	4.756	5.025	5.245	5.433	5.595	5.739	5.867	5.983	6.088	6.186	6.275
10	3.151	3.874	4.327	4.655	4.913	5.125	5.305	5.461	5.598	5.722	5.833	5.934	6.027	6.114
11	3.113	3.817	4.257	4.574	4.824	5.029	5.203	5.353	5.486	5.605	5.713	5.811	5.901	5.984
12	3.081	3.771	4.199	4.509	4.751	4.950	5.119	5.266	5.395	5.510	5.615	5.710	5.797	5.878
13	3.055	3.732	4.151	4.454	4.690	4.885	5.050	5.192	5.318	5.431	5.533	5.625	5.710	5.789
14	3.033	3.700	4.111	4.407	4.639	4.829	4.991	5.130	5.253	5.364	5.463	5.554	5.637	5.714
15	3.014	3.672	4.076	4.368	4.595	4.782	4.940	5.077	5.198	5.306	5.403	5.492	5.574	5.649
16	2.998	3.648	4.046	4.333	4.557	4.741	4.897	5.031	5.150	5.256	5.352	5.439	5.519	5.593
17	2.984	3.627	4.020	4.303	4.524	4.705	4.858	4.991	5.108	5.212	5.306	5.392	5.471	5.544

Q-Cutoffs Table ($\alpha = 0.025$)

($\alpha = \mathbf{0.025}$) **GOSSET'S Q-CUTOFFS,** $q^*_{M,\nu;\alpha}$ $\mathbb{P}(Q > q^*_{M,\nu;\alpha}) = \alpha$

ν \ M	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	5.907	7.654	8.811	9.663	10.34	10.89	11.36	11.77	12.13	12.45	12.74	13.01	13.25	13.47
4	4.943	6.234	7.090	7.719	8.216	8.627	8.977	9.280	9.548	9.787	10.00	10.20	10.38	10.55
5	4.474	5.551	6.258	6.777	7.188	7.528	7.817	8.069	8.291	8.489	8.669	8.833	8.984	9.123
6	4.198	5.152	5.773	6.228	6.588	6.886	7.139	7.360	7.554	7.729	7.886	8.030	8.162	8.285
7	4.018	4.891	5.456	5.869	6.196	6.466	6.695	6.895	7.072	7.230	7.373	7.503	7.623	7.734
8	3.891	4.710	5.234	5.617	5.920	6.170	6.383	6.568	6.732	6.878	7.011	7.132	7.243	7.346
9	3.797	4.575	5.070	5.431	5.716	5.951	6.151	6.326	6.479	6.617	6.742	6.856	6.960	7.057
10	3.725	4.471	4.943	5.287	5.559	5.783	5.973	6.138	6.284	6.415	6.534	6.642	6.742	6.834
11	3.667	4.389	4.843	5.174	5.434	5.649	5.831	5.990	6.130	6.255	6.369	6.472	6.568	6.656
12	3.620	4.322	4.761	5.081	5.333	5.540	5.716	5.869	6.004	6.125	6.234	6.334	6.426	6.511
13	3.582	4.267	4.694	5.005	5.249	5.450	5.620	5.769	5.900	6.017	6.123	6.220	6.308	6.391
14	3.549	4.220	4.638	4.941	5.178	5.374	5.540	5.684	5.812	5.926	6.029	6.123	6.209	6.289
15	3.521	4.180	4.589	4.886	5.118	5.309	5.471	5.612	5.737	5.848	5.948	6.040	6.125	6.203
16	3.497	4.147	4.548	4.838	5.066	5.253	5.412	5.550	5.671	5.780	5.879	5.969	6.051	6.127
17	3.476	4.116	4.512	4.797	5.021	5.204	5.360	5.496	5.615	5.722	5.818	5.906	5.987	6.062

Q-Cutoffs Table $(\alpha = 0.01)$

$(\alpha = \mathbf{0.01})$ **GOSSET'S Q-CUTOFFS,** $q^*_{M,\nu;\alpha}$ $\mathbb{P}(Q > q^*_{M,\nu;\alpha}) = \alpha$

ν \ M	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
3	8.260	10.61	12.17	13.33	14.25	15.00	15.65	16.20	16.69	17.13	17.53	17.89	18.22	18.52
4	6.511	8.110	9.175	9.961	10.59	11.10	11.54	11.92	12.26	12.56	12.84	13.09	13.32	13.53
5	5.702	6.966	7.806	8.425	8.917	9.324	9.671	9.973	10.24	10.48	10.70	10.89	11.07	11.24
6	5.243	6.325	7.035	7.559	7.975	8.319	8.613	8.869	9.096	9.299	9.483	9.651	9.806	9.949
7	4.949	5.912	6.543	7.007	7.375	7.680	7.941	8.167	8.368	8.548	8.711	8.860	8.997	9.123
8	4.745	5.630	6.204	6.626	6.961	7.238	7.475	7.681	7.863	8.027	8.175	8.311	8.436	8.551
9	4.596	5.424	5.957	6.349	6.659	6.916	7.134	7.325	7.494	7.646	7.783	7.909	8.025	8.132
10	4.482	5.267	5.769	6.137	6.429	6.670	6.876	7.055	7.213	7.356	7.485	7.603	7.711	7.812
11	4.392	5.143	5.621	5.971	6.248	6.477	6.672	6.842	6.992	7.127	7.249	7.361	7.464	7.559
12	4.320	5.043	5.502	5.837	6.102	6.321	6.508	6.670	6.814	6.943	7.060	7.166	7.265	7.355
13	4.260	4.961	5.404	5.727	5.982	6.193	6.372	6.528	6.667	6.791	6.903	7.005	7.100	7.187
14	4.210	4.893	5.322	5.635	5.882	6.085	6.259	6.410	6.543	6.663	6.771	6.871	6.962	7.046
15	4.167	4.834	5.252	5.557	5.796	5.994	6.163	6.309	6.439	6.555	6.660	6.756	6.844	6.926
16	4.131	4.784	5.192	5.489	5.723	5.916	6.080	6.222	6.348	6.461	6.564	6.657	6.743	6.823
17	4.099	4.740	5.140	5.431	5.659	5.848	6.008	6.147	6.270	6.380	6.480	6.571	6.655	6.733

Q-Cutoffs Table $(\alpha = 0.005)$

$(\alpha = \mathbf{0.005})$ **GOSSET'S Q-CUTOFFS,** $q^*_{M,\nu;\alpha}$ $\mathbb{P}(Q > q^*_{M,\nu;\alpha}) = \alpha$

ν \ M	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
3	10.54	13.49	15.46	16.91	18.07	19.02	19.83	20.53	21.15	21.70	22.20	22.66	23.07	23.46
4	7.916	9.800	11.06	12.00	12.74	13.36	13.88	14.33	14.74	15.10	15.42	15.72	15.99	16.24
5	6.751	8.183	9.142	9.850	10.41	10.88	11.28	11.63	11.93	12.21	12.46	12.69	12.90	13.09
6	6.105	7.298	8.089	8.673	9.137	9.523	9.852	10.14	10.39	10.62	10.83	11.02	11.19	11.35
7	5.698	6.742	7.430	7.938	8.342	8.677	8.963	9.213	9.434	9.632	9.812	9.976	10.13	10.27
8	5.420	6.364	6.981	7.437	7.799	8.099	8.356	8.579	8.777	8.955	9.117	9.264	9.400	9.526
9	5.218	6.090	6.657	7.075	7.407	7.682	7.916	8.121	8.303	8.466	8.614	8.749	8.873	8.989
10	5.065	5.884	6.413	6.802	7.110	7.366	7.584	7.775	7.944	8.096	8.233	8.359	8.475	8.582
11	4.945	5.722	6.222	6.588	6.879	7.120	7.326	7.505	7.664	7.807	7.936	8.054	8.163	8.264
12	4.849	5.593	6.068	6.417	6.694	6.923	7.118	7.288	7.439	7.574	7.697	7.810	7.913	8.009
13	4.769	5.487	5.943	6.277	6.542	6.761	6.947	7.110	7.255	7.384	7.501	7.609	7.708	7.799
14	4.703	5.398	5.839	6.161	6.415	6.626	6.806	6.962	7.101	7.225	7.338	7.441	7.536	7.625
15	4.647	5.323	5.750	6.062	6.308	6.512	6.685	6.837	6.970	7.091	7.200	7.299	7.391	7.476
16	4.599	5.259	5.674	5.977	6.216	6.414	6.582	6.729	6.859	6.975	7.081	7.178	7.267	7.349
17	4.557	5.203	5.609	5.904	6.137	6.329	6.493	6.636	6.762	6.875	6.978	7.072	7.158	7.239

PART II:

Finding Significantly Different Pop. Means in 1F bcrANOVA:

Tukey Complete Pairwise Post-Hoc Comparison

Simultaneous Q-CI's for Mean Differences

Suppose a 1F bcrANOVA results in the rejection of null hypothesis H_0^A. Then, at least two of the population means significantly differ, but ANOVA does not indicate which means significantly differ.

Thus, a post-hoc procedure must be used to find significant differences:

Proposition

Given an experiment with I groups each of size J such that the 1F bcrANOVA assumptions are satisfied.

Then the simultaneous $100(1-\alpha)\%$ Q-CI's for all mean differences $\mu_i - \mu_j$ are:

$$(\overline{x}_{i\bullet} - \overline{x}_{j\bullet}) \pm q^*_{I,\nu_{res};\alpha} \cdot \sqrt{MS_{err}/J} \qquad \forall i < j \qquad [\nu_{res} := I(J-1)]$$

If Q-CI for $\mu_i - \mu_j$ does not contain zero, then μ_i & μ_j significantly differ.

Unfortunately, computing <u>all</u> the Q-CI's is tedious and wasteful.

Tukey Complete Pairwise Post-Hoc Comparison

Fortunately, the following procedure is far more efficient:

Proposition

Given an experiment with I groups each of size J $\quad [\nu_{res} := I(J-1)]$
where 1F bcrANOVA rejects H_0^A at significance level α.
Then, to determine which population means significantly differ:

1. *Compute significant difference width* $w = q^*_{I,\nu_{res};\alpha} \cdot \sqrt{MS_{res}/J}$
2. *Sort the group means in ascending order:* $\bar{x}_{(1)\bullet} \leq \bar{x}_{(2)\bullet} \leq \cdots \leq \bar{x}_{(I)\bullet}$
3. *For each sorted group mean $\bar{x}_{(k)\bullet}$:*
    - *If $\bar{x}_{(k+1)\bullet} \notin \left[\bar{x}_{(k)\bullet},\ \bar{x}_{(k)\bullet} + w\right]$, repeat STEP 3 with next sorted mean.*
    - *Else, underline $\bar{x}_{(k)\bullet}$ and all larger means within a distance of w with new line.*

Interpretation:

- *Group means sharing a common underline implies they are not significantly different from one another.*
- *Group means not sharing a common underline implies they are significantly different from one another.*

CI's for Comparing Collections of Group Means

Proposition

Given an experiment with I groups each of size J $\quad [\nu_{res} := I(J-1)]$
such that the 1F bcrANOVA assumptions are satisfied.
Let constants $c_1, c_2, \cdots, c_I \in \mathbb{R}$ such that they sum to zero: $\quad \sum_i c_i = 0.$

Then the $100(1-\alpha)\%$ t-CI for mean collection difference $\sum_i c_i \mu_i$ is:

$$\left(\sum_i c_i \bar{x}_{i\bullet}\right) \pm t^*_{\nu_{res};\alpha/2} \cdot \sqrt{MS_{res} \cdot \sum_i c_i^2 / J}$$

Examples of collection differences:

# **GROUPS:**	**COLLECTIONS TO COMPARE:**	$\sum_i c_i \mu_i$
$I = 3$	μ_1 VS. (μ_2, μ_3)	$\mu_1 - \frac{1}{2}(\mu_2 + \mu_3)$
$I = 3$	(μ_1, μ_2) VS. μ_3	$\frac{1}{2}(\mu_1 + \mu_2) - \mu_3$
$I = 3$	(μ_1, μ_3) VS. μ_2	$\frac{1}{2}(\mu_1 + \mu_3) - \mu_2$
$I = 4$	μ_1 VS. (μ_2, μ_3, μ_4)	$\mu_1 - \frac{1}{3}(\mu_2 + \mu_3 + \mu_4)$
$I = 4$	(μ_1, μ_2) VS. (μ_3, μ_4)	$\frac{1}{2}(\mu_1 + \mu_2) - \frac{1}{2}(\mu_3 + \mu_4)$
$I = 4$	(μ_1, μ_2, μ_3) VS. μ_4	$\frac{1}{3}(\mu_1 + \mu_2 + \mu_3) - \mu_4$

Other Post-Hoc Comparisons

METHOD:	YEAR:	WORKS FOR UNEQUAL SIZES?	WORKS FOR UNEQUAL VARIANCES?
Bonferroni-Dunn	1958	NO	NO
Tukey	1953	NO	NO
Tukey-Kramer	1956	YES	NO
Fisher	1949	YES	NO
Fisher-Hayter	1986	YES	NO
Scheffé	1953	YES	NO
Kaiser-Bowden	1983	YES	YES

The Bonferroni-Dunn post-hoc test is very conservative as it is simply a set of t-tests with a correction to the α significance levels.

Only Tukey & Tukey-Kramer post-hoc comparisons will be considered.

Miller's 1966 textbook♠ was the first to survey many of these techniques.

♠ R.G. Miller, *Simultaneous Statistical Inference*, Springer, 1966.

Planned Comparisons

Planned comparisons are used when particular group mean comparisons are desired even before the ANOVA is performed.

Often, these involve comparing treatment groups to a **control group**, which is typical in product brand intervention studies.

The following methods are specifically tailored for control groups:

METHOD:	**YEAR:**	**WORKS FOR UNEQUAL SIZES?**	**WORKS FOR UNEQUAL VARIANCES?**
Dunnett	1955	**YES**	**NO**
Dunnett T3	1980	**YES**	**YES**
Dunnett C	1980	**YES**	**YES**

Planned comparisons will not be considered at all in this course.

References

[GW]	F.J. Gravetter L.B. Wallnau	*Statistics for the Behavioral Sciences*	7^{th} Ed	2007
[LH]	R.G. Lomax D.L. Hahs-Vaughn	*Statistical Concepts : A Second Course*	4^{th} Ed	2012
[S]	J.P. Stevens	*Intermediate Statistics A Modern Approach*	3^{rd} Ed	2007

Textbook Logistics for Section 10.2

TEXTBOOK TERMINOLOGY:	SLIDES/OUTLINE TERMINOLOGY:
Treatment/Cell	Group
Multiple Comparisons	Post-Hoc Comparisons

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Expected Value	$E(X)$	$\mathbb{E}[X]$
Variance	$V(X)$	$\mathbb{V}[X]$
Sum of Squares of Factor A	SSTr	SS_A
Mean Square of Factor A	MSTr	MS_A
Sum of Squares of Residuals	SSE	SS_{res}
Mean Square of Residuals	MSE	MS_{res}
Null Hypothesis for Factor A	H_0	H_0^A
Alt. Hypothesis for Factor A	H_A	H_A^A
Q-Cutoff	$Q_{\alpha,m,\nu}$	$q^*_{M,\nu;\alpha}$

Fin.