1F Unbalanced Completely Randomized ANOVA

Engineering Statistics II
Section 10.3

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PART I:

1-Factor Fixed Effects Linear (Statistical) Models:

Definitions, Examples

Least Squares Estimators (LSE's)

Best Linear Unbiased Estimators (BLUE's)

Gauss-Markov Theorem

1-Factor Unbalanced Fixed Effects Linear Models

With many-sample inference, it's convenient to use a **linear model**:

Definition

(1-Factor Unbalanced Fixed Effects Linear Model)

Given a 1-factor unbalanced experiment with $I > 2$ groups, each of size J_i.

Let $X_{ij} \equiv$ random variable for j^{th} measurement in the i^{th} group.

Then, the **unbalanced fixed effects linear model** for the experiment is:

$$X_{ij} = \mu + \alpha_i^A + E_{ij} \quad \text{where} \quad E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$$

where:

$$\begin{aligned} \mu &\equiv \text{population grand mean of all } I \text{ population means} \\ \alpha_i^A &\equiv \text{deviation of } i^{th} \text{ population mean } \mu_i \text{ from } \mu \text{ due to Factor A} \\ E_{ij} &\equiv \text{rv for error/noise applied to } j^{th} \text{ measurement in } i^{th} \text{ group} \end{aligned}$$

Fixed effects means all relevant levels of factor A are considered in model.

$$X_{ij} = \mu$$

$$\mu := 3.2$$

$$\mu_1 = 3.2,\ \mu_2 = 3.2,\ \mu_3 = 3.2$$

FACTOR A:	**MEASUREMENTS:**			
Level 1 $(x_{1\bullet})$	$x_{11} = 3.2,$	$x_{12} = 3.2,$	$x_{13} = 3.2,$	
Level 2 $(x_{2\bullet})$	$x_{21} = 3.2,$	$x_{22} = 3.2,$	$x_{23} = 3.2,$	$x_{24} = 3.2$
Level 3 $(x_{3\bullet})$	$x_{31} = 3.2,$	$x_{32} = 3.2,$		

1F Unbalanced Linear Models (Motivating Example)

$$X_{ij} = \mu + \alpha_i^A$$

$$\mu := 3.2$$

$$\alpha_1^A := -5.5,\ \alpha_2^A := -2.0,\ \alpha_3^A := 7.5$$

$$\mu_1 = -2.3,\ \mu_2 = 1.2,\ \mu_3 = 10.7$$

FACTOR A:	**MEASUREMENTS:**			
Level 1 ($x_{1\bullet}$)	$x_{11} = -2.3$,	$x_{12} = -2.3$,	$x_{13} = -2.3$,	
Level 2 ($x_{2\bullet}$)	$x_{21} = 1.2$,	$x_{22} = 1.2$,	$x_{23} = 1.2$,	$x_{24} = 1.2$
Level 3 ($x_{3\bullet}$)	$x_{31} = 10.7$,	$x_{32} = 10.7$,		

1F Unbalanced Linear Models (Motivating Example)

$$X_{ij} = \mu + \alpha_i^A + E_{ij}$$

$$\mu := 3.2$$

$$\alpha_1^A := -5.5,\ \alpha_2^A := -2.0,\ \alpha_3^A := 7.5$$

$$\mu_1 = -2.3,\ \mu_2 = 1.2,\ \mu_3 = 10.7$$

$$E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2 := 3.24)$$

FACTOR A:	**MEASUREMENTS:**			
Level 1 ($x_{1\bullet}$)	$x_{11} = -1.23$,	$x_{12} = -1.17$,	$x_{13} = 0.05$,	
Level 2 ($x_{2\bullet}$)	$x_{21} = 0.54$,	$x_{22} = 1.03$,	$x_{23} = 0.62$,	$x_{24} = 1.63$
Level 3 ($x_{3\bullet}$)	$x_{31} = 13.64$,	$x_{32} = 12.30$,		

1-Factor Linear Models (Least-Squares Estimators)

Like all population parameters, linear model parameters can be estimated:

Proposition

Given a 1-factor unbalanced linear model: (i^{th} *group has* J_i *measurements)*

$$X_{ij} = \mu + \alpha_i^A + E_{ij} \quad \textit{where} \quad E_{ij} \overset{iid}{\sim} \textit{Normal}(0, \sigma^2)$$

Then:

(a) The ***least-squares♠♣ estimators (LSE's)*** *for the model parameters are:*

$$\begin{aligned} \hat{\mu} &= \bar{x}_{\bullet\bullet} \\ \hat{\alpha}_i^A &= \bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet} \end{aligned} \quad \textit{where} \quad \begin{aligned} \bar{x}_{\bullet\bullet} &\equiv \textit{Grand sample mean} \\ \bar{x}_{i\bullet} &\equiv \textit{Sample mean of } i^{th} \textit{ group} \end{aligned}$$

(b) For these least-squares estimators, it's required that $\sum_i J_i \hat{\alpha}_i^A = 0$.

(c) These least-squares estimators are all unbiased.

PROOF: The general case is left as an ungraded exercise for the reader.

♠A.M. Legendre, *Nouvelles Méthodes pour la Détermination des Orbites des Comètes*, 1806.

♣Gauss, *Theoria Motus Corporum Coelestrium in Sectionibus Conicis Solem Ambientium*, 1809.

With the model parameter estimators in hand, responses can be predicted:

Definition

(Predicted Responses)

Given a 1-factor unbalanced linear model: (i^{th} group has J_i measurements)

$$X_{ij} = \mu + \alpha_i^A + E_{ij} \quad \text{where} \quad E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$$

Then the corresponding **predicted responses**, denoted $\hat{x}_{ij}$, are:

$$\hat{x}_{ij} := \hat{\mu} + \hat{\alpha}_i^A = \bar{x}_{\bullet\bullet} + (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet}) = \bar{x}_{i\bullet}$$

SYNONYMS: Predicted values, fitted values

1-Factor Linear Models (Residuals)

With the predicted responses in hand, residuals can be computed:

Definition

(Residuals)

Given a 1-factor unbalanced linear model: (i^{th} group has J_i measurements)

$$X_{ij} = \mu + \alpha_i^A + E_{ij} \quad \text{where} \quad E_{ij} \overset{iid}{\sim} \text{Normal}(0, \sigma^2)$$

Then the corresponding predicted responses, denoted $\hat{x}_{ij}$, are:

$$\hat{x}_{ij} := \hat{\mu} + \hat{\alpha}_i^A = \overline{x}_{\bullet\bullet} + (\overline{x}_{i\bullet} - \overline{x}_{\bullet\bullet}) = \overline{x}_{i\bullet}$$

Moreover, the corresponding **residuals**, denoted x_{ij}^{res}, are:

$$x_{ij}^{res} := x_{ij} - \hat{x}_{ij} = x_{ij} - \overline{x}_{i\bullet}$$

Point estimators for a linear model should be ideal ones:

Definition

(Best Linear Unbiased Estimators – BLUE's)

A point estimator $\hat{\theta}$ is called a **best linear unbiased estimator (BLUE)** if:

- It estimates a parameter θ of a linear model.
- It is a linear combination of the data points: $\hat{\theta} := \sum_{k=1}^{n} c_k x_k$
- It is an unbiased estimator: $\mathbb{E}[\hat{\theta}] = \theta$
- It has minimum variance of all such unbiased estimators.

REMARK: BLUE's are generally easier to construct & prove than UMVUE's.

For a 1-factor linear model: $X_{ij} = \mu + \alpha_i^A + E_{ij}$

$\hat{\mu}, \hat{\alpha}_i^A$ are each linear combinations of data points in the linear model.
A particular example of demonstrating this is done in EX 10.1.1.

1-Factor Linear Models (Gauss-Markov Theorem)

Ideally, point estimators for linear model parameters should be BLUE's:

Theorem

(Gauss[1]-Markov[2] Theorem)

Given a 1-factor unbalanced linear model: (i^{th} group has J_i measurements)

$$X_{ij} = \mu + \alpha_i^A + E_{ij}$$

Moreover, suppose the following conditions are all satisfied:

$$\begin{aligned} \mathbb{E}[E_{ij}] &= 0 && (\textit{errors are all centered at zero}) \\ \mathbb{V}[E_{ij}] &= \sigma^2 && (\textit{errors all have the same finite variance}) \\ \mathbb{C}[E_{ij}, E_{i'j'}] &= 0 && (\textit{errors are uncorrelated when } i \neq i' \textit{ or } j \neq j') \end{aligned}$$

Then, the least-squares estimators (LSE's) $\hat{\mu}, \hat{\alpha}_i^A$ are all BLUE's.

PROOF: Omitted due to time.

[1] C.F. Gauss, "Theoria Combinationis Observationum Erroribus Minimis Obnoxiae", (1823), 1-58.

[2] A.A. Markov, *Calculus of Probabilities*, 1^{st} Edition, 1900.

PART II:

1-Factor Unbalanced Completely Randomized ANOVA (1F ucrANOVA)

1F ucrANOVA (Motivation)

A 1F ucrANOVA is used if:

- Some experimental units (EU's) in a balanced experiment...
  - (if machines) ...malfunction, lose power, become damaged, are stolen or die.
  - (if plants) ...become ill, are infested with parasites, are stolen or die.
  - (if animals) ...become ill, bite experimenters†, are stolen or die.
  - (if people) ...move away, do not show up, fail to comply, become ill or die.
- The levels of Factor A naturally differ in size – e.g. classroom rosters†.
- Some levels of Factor A are prohibitively expensive to carry out‡...
  - ...and, hence, have fewer EU's.
- Some levels of Factor A are far more interesting than others‡...
  - ...and, hence, have more EU's.

†D.C. Howell, *Statistical Methods for Psychology*, 7^{th} Edition, Cengage, 2010. (§15.2)
‡D.C. Montgomery, *Design and Analysis of Experiments*, 7^{th} Edition, Wiley, 2009. (§11.7)

1-Factor Unbalanced Completely Randomized Design

An example **unbalanced completely randomized design** entails:

- Collect 9 relevant experimental units (EU's): $\text{EU}_1, \text{EU}_2, \cdots, \text{EU}_9$
- Produce a random shuffle sequence using software:
  $(4, 7, 9;\ 5, 3, 6, 2;\ 1, 8)$
- Use random shuffle sequence to assign the EU's into the I levels:

FACTOR A:	MEASUREMENTS:			
Level 1	EU_4,	EU_7,	EU_9	
Level 2	EU_5,	EU_3,	EU_6,	EU_2
Level 3	EU_1,	EU_8		

- Measure each EU appropriately (note the change in notation):

FACTOR A:	MEASUREMENTS:			
Level 1 $(x_{1\bullet})$	x_{11},	x_{12},	x_{13}	
Level 2 $(x_{2\bullet})$	x_{21},	x_{22},	x_{23},	x_{24}
Level 3 $(x_{3\bullet})$	x_{31},	x_{32}		

$$\begin{aligned} \text{EU}_k &\equiv \left(k^{th} \text{ experimental unit collected}\right) \\ x_{ij} &\equiv \left(\text{Measurement of } j^{th} \text{ experimental unit in } i^{th} \text{ level}\right) \\ x_{i\bullet} &\equiv \left(\text{Group of all measurements in } i^{th} \text{ level}\right) \end{aligned}$$

How to Produce Random Shuffle Sequence

How to produce random shuffle sequence of numbers 1 through N:

LANGUAGE:	MINIMUM CODE:
Matlab	`s=1:N;` `s(randperm(length(s)))`
Python	`import random` `random.sample(range(1,N+1),N)`
R	`sample(N)`

1F ucrANOVA (Fixed Effects Model Assumptions)

Proposition

(1F ucrANOVA Fixed Effects Model Assumptions)

- *(**1 Desired Factor**) Factor A has I levels.*
- *(**All Factor Levels are Considered**) AKA Fixed Effects.*
- *(**Replication in Groups**) Each group has $J_i > 1$ units.*
- *(**Distinct Exp. Units**) All $\sum_i J_i$ units are distinct from each other.*

- *(**Random Assignment across Groups**)*

- *(**Independence**) All measurements on units are independent.*
- *(**Normality**) All groups are approximately normally distributed.*
- *(**Equal Variances**) All groups have approximately same variance.*

Mnemonic: ***1DF AFLaC RiG DEU | RAaG | I.N.EV***

1F ucrANOVA Fixed Effects Linear Model

Fixed effects means all relevant levels of factor A are considered in model.

1F ucrANOVA Fixed Effects Linear Model

I	$\equiv$	# groups to compare
J_i	$\equiv$	# measurements in i^{th} group
X_{ij}	$\equiv$	rv for j^{th} measurement taken from i^{th} group
μ_i	$\equiv$	Mean of i^{th} population or true average response from i^{th} group
μ	$\equiv$	Common population mean or true average overall response
α_i^A	$\equiv$	Deviation from μ due to i^{th} group
E_{ij}	$\equiv$	Deviation from μ due to random error

ASSUMPTIONS: $E_{ij} \overset{iid}{\sim}$ Normal $(0, \sigma^2)$

$$X_{ij} = \mu + \alpha_i^A + E_{ij} \quad \text{where} \quad \sum_i J_i \alpha_i^A = 0$$

$$H_0^A : \quad \text{All} \quad \alpha_i^A = 0$$
$$H_A^A : \quad \text{Some} \quad \alpha_i^A \neq 0$$

1F ucrANOVA (Sums of Squares "Partition" Variation)

$$\underbrace{\text{SS}_{total}}_{Total\ Variation\ in\ Experiment} = \underbrace{\text{SS}_{A}}_{Variation\ due\ to\ Factor\ A} + \underbrace{\text{SS}_{res}}_{Unexplained\ Variation}$$

$$\sum_{ij}(x_{ij} - \hat{\mu})^2 = \sum_{ij}(\hat{\alpha}_i^A)^2 + \sum_{ij}(x_{ij}^{res})^2$$

$$\sum_i \sum_{j=1}^{J_i}(x_{ij} - \bar{x}_{\bullet\bullet})^2 = \sum_i \sum_{j=1}^{J_i}(\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2 + \sum_i \sum_{j=1}^{J_i}(x_{ij} - \bar{x}_{i\bullet})^2$$

$$\underbrace{\nu}_{Total\ dof's\ in\ Experiment} = \underbrace{\nu_A}_{'Between\ Groups'\ dof's} + \underbrace{\nu_{res}}_{'Within\ Groups'\ dof's}$$

$$\nu = n - 1 \quad \left(n := \sum_i J_i\right) \qquad \nu_A = I - 1 \qquad \nu_{res} = n - I$$

1F ucrANOVA F-Test & Table (Given x_{ij})

1. Determine df's: $n := \sum_i J_i, \quad \nu_A = I - 1, \quad \nu_{res} = n - I$
2. Compute Group Means: $\bar{x}_{i\bullet} := \frac{1}{J_i} \sum_{j=1}^{J_i} x_{ij}$
3. Compute Grand Mean: $\bar{x}_{\bullet\bullet} := \frac{1}{I} \sum_i \bar{x}_{i\bullet}$
4. Compute $\text{SS}_{res} := \sum_i \sum_{j=1}^{J_i} (x_{ij}^{res})^2 = \sum_i \sum_{j=1}^{J_i} (x_{ij} - \bar{x}_{i\bullet})^2$ and $\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}}$
5. Compute $\text{SS}_A := \sum_i \sum_{j=1}^{J_i} (\hat{\alpha}_i^A)^2 = \sum_i \sum_{j=1}^{J_i} (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2$ and $\text{MS}_A := \frac{\text{SS}_A}{\nu_A}$
6. Compute Test Statistic Value: $f_A = \frac{\text{MS}_A}{\text{MS}_{res}}$
7. Compute P-value: $p_A := \mathbb{P}(F > f_A) \approx 1 - \Phi_F(f_A; \nu_A, \nu_{res})$
8. Render Decision:

(by SW)	If	$p_A \leq \alpha$	then reject H_0^A for H_A^A, else accept H_0^A.
(by hand)	If	$f_A \geq f^*_{\nu_A, \nu_{res}; \alpha}$	then reject H_0^A for H_A^A, else accept H_0^A.

1F ucrANOVA F-Test & Table (Given $\bar{x}_{i\bullet}$ & s_i^2)

1. Determine df's: $n := \sum_i J_i, \quad \nu_A = I - 1, \quad \nu_{res} = n - I$
2. Compute Grand Mean: $\bar{x}_{\bullet\bullet} := \frac{1}{I} \sum_i \bar{x}_{i\bullet}$
3. Compute $\text{SS}_{res} := \sum_i \sum_{j=1}^{J_i} (x_{ij}^{res})^2 = \sum_i (J_i - 1) \cdot s_i^2$ and $\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}}$
4. Compute $\text{SS}_A := \sum_i \sum_{j=1}^{J_i} (\hat{\alpha}_i^A)^2 = \sum_i \sum_{j=1}^{J_i} (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2$ and $\text{MS}_A := \frac{\text{SS}_A}{\nu_A}$
5. Compute Test Statistic Value: $f_A = \frac{\text{MS}_A}{\text{MS}_{res}}$
6. Compute P-value: $p_A := \mathbb{P}(F > f_A) \approx 1 - \Phi_F(f_A; \nu_A, \nu_{res})$
7. Render Decision:

   (by SW) If $p_A \leq \alpha$ then reject H_0^A for H_A^A, else accept H_0^A.

   (by hand) If $f_A \geq f^*_{\nu_A, \nu_{res}; \alpha}$ then reject H_0^A for H_A^A, else accept H_0^A.

1F ucrANOVA F-Test & Table (Given $\bar{x}_{i\bullet}$ & $\widehat{\sigma}_{\bar{x}_{i\bullet}}$)

1. Determine df's: $n := \sum_i J_i, \quad \nu_A = I - 1, \quad \nu_{res} = n - I$
2. Compute Group Variances: $s_i^2 = \sqrt{J_i} \cdot \widehat{\sigma}_{\bar{x}_{i\bullet}}$
3. Compute Grand Mean: $\bar{x}_{\bullet\bullet} := \frac{1}{I} \sum_i \bar{x}_{i\bullet}$
4. Compute $\text{SS}_{res} := \sum_i \sum_{j=1}^{J_i} (x_{ij}^{res})^2 = \sum_i (J_i - 1) \cdot s_i^2$ and $\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}}$
5. Compute $\text{SS}_A := \sum_i \sum_{j=1}^{J_i} (\hat{\alpha}_i^A)^2 = \sum_i \sum_{j=1}^{J_i} (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2$ and $\text{MS}_A := \frac{\text{SS}_A}{\nu_A}$
6. Compute Test Statistic Value: $f_A = \frac{\text{MS}_A}{\text{MS}_{res}}$
7. Compute P-value: $p_A := \mathbb{P}(F > f_A) \approx 1 - \Phi_F(f_A; \nu_A, \nu_{res})$
8. Render Decision:

(by SW)	If	$p_A \le \alpha$	then reject H_0^A for H_A^A, else accept H_0^A.
(by hand)	If	$f_A \ge f^*_{\nu_A, \nu_{res}; \alpha}$	then reject H_0^A for H_A^A, else accept H_0^A.

1F ucrANOVA Table (α) (Significance Level α)						
Variation Source	**df**	**Sum of Squares**	**Mean Square**	**F Stat Value**	**P-value**	**Decision**
Factor A	ν_A	SS_A	MS_A	f_A	p_A	Acc/Rej H_0^A
Unknown	ν_{res}	SS_{res}	MS_{res}			
Total	ν	SS_{total}				

1F ucrANOVA (Expected Mean Squares)

Proposition

Given 1-factor experiment satisfying the 1F ucrANOVA assumptions. Then:

$$\begin{aligned}
(i) \quad \mathbb{E}[MS_{res}] &= \sigma^2 \\
(ii) \quad \mathbb{E}[MS_A] &= \sigma^2 + \frac{1}{I-1}\sum_i J_i(\alpha_i^A)^2
\end{aligned}$$

PROOF: Omitted as it's similar (but a bit more tedious) to 1F bcrANOVA.

1F ucrANOVA (Point Estimators of σ^2)

Proposition

(Point Estimation of Mean Squares)

Given 1F balanced experiment satisfying 1F ucrANOVA assumptions. Then:

(i) MS_{res} *is always an unbiased point estimator of the population variance:*

$$H_0 \text{ is indeed true OR } H_0 \text{ is indeed false} \implies \mathbb{E}[MS_{res}] = \sigma^2$$

(ii) If the status quo prevails, MS_A *is an unbiased estimator of pop. variance:*

$$H_0 \text{ is indeed true} \implies \mathbb{E}[MS_A] = \sigma^2$$

(iii) If the status quo fails, MS_A *tends to overestimate population variance:*

$$H_0 \text{ is indeed false} \implies \mathbb{E}[MS_A] > \sigma^2$$

PROOF: Omitted as it's similar (but a bit more tedious) to 1F bcrANOVA.

Simultaneous Q-CI's for Mean Differences

Suppose a 1F ucrANOVA results in the rejection of null hypothesis H_0^A.

Then, at least two of the population means significantly differ, but ANOVA does not indicate which means significantly differ.

Therefore, a post-hoc procedure must be used:

Proposition

Given an experiment with I groups each of size J_i $(n := \sum_i J_i)$ such that the 1F ucrANOVA assumptions are satisfied.

Then the approximate simultaneous $100(1-\alpha)\%$ Q-CI's for all mean differences $\mu_i - \mu_j$ are:

$$(\bar{x}_{i\bullet} - \bar{x}_{j\bullet}) \pm q^*_{I,\nu_{res};\alpha} \cdot \sqrt{MS_{res} \cdot \frac{1}{2}\left(\frac{1}{J_i} + \frac{1}{J_j}\right)} \qquad \forall i < j \qquad (\nu_{res} := n - I)$$

If Q-CI for $\mu_i - \mu_j$ does not contain zero, then μ_i & μ_j significantly differ.

Unfortunately, computing all the Q-CI's is tedious and wasteful.

Tukey-Kramer Complete Pairwise Post-Hoc Comp.

Fortunately, the following procedure is far more efficient:

Proposition

Given an experiment with I groups each of size J_i $(n := \sum_i J_i,\ \nu_{res} := n - I)$ where 1F ucrANOVA rejects H_0^A at level α and the J_i's only differ slightly. Then, to determine which population means significantly differ:

1. *Sort the group means in ascending order:* $\bar{x}_{(1)\bullet} \leq \bar{x}_{(2)\bullet} \leq \cdots \leq \bar{x}_{(I)\bullet}$
2. *Find significant difference widths* $w_{(ij)} = q^*_{I,\nu_{res};\alpha} \cdot \sqrt{MS_{res} \cdot \frac{1}{2}\left(\frac{1}{J_{(i)}} + \frac{1}{J_{(j)}}\right)}$
3. *If* $\bar{x}_{(j)\bullet} \in \left[\bar{x}_{(i)\bullet},\ \bar{x}_{(i)\bullet} + w_{(ij)}\right]$*, underline* $\bar{x}_{(i)\bullet}$ *and* $\bar{x}_{(j)\bullet}$ *with new line.*
4. *Repeat STEP 1 with all sorted mean pairs* $\bar{x}_{(i)\bullet}, \bar{x}_{(j)\bullet}$ *such that* $i < j$.

Interpretation:

- *Group means sharing a common underline implies they are not significantly different from one another.*
- *Group means not sharing a common underline implies they are significantly different from one another.*

PART III:

1-Factor ANOVA Model (Adequacy) Checking

Standardized Residuals

Checking for Outliers

Checking Normality Assumption

Checking Independence Assumption

Checking Equal Variances Assumption

1F ANOVA Model Checking: Standardized Residuals

Definition

(Standardized Residuals)

Given a 1-factor experiment, either balanced or only slightly unbalanced:

$$X_{ij} = \mu + \alpha_i^A + E_{ij}$$

Moreover, suppose 1F bcrANOVA / ucrANOVA was performed accordingly.

Then, the **standardized residuals**† are defined to be:

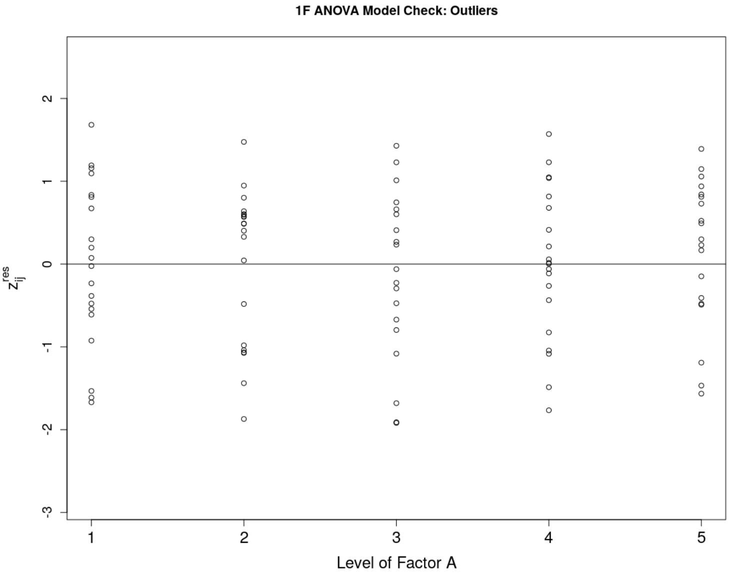
$$z_{ij}^{res} := \frac{x_{ij}^{res}}{\sqrt{\text{SS}_{res}/(n-1)}}$$

An alternative definition‡ that's reasonable but not used here is: $\dfrac{x_{ij}^{res}}{\sqrt{\text{MS}_{res}}}$

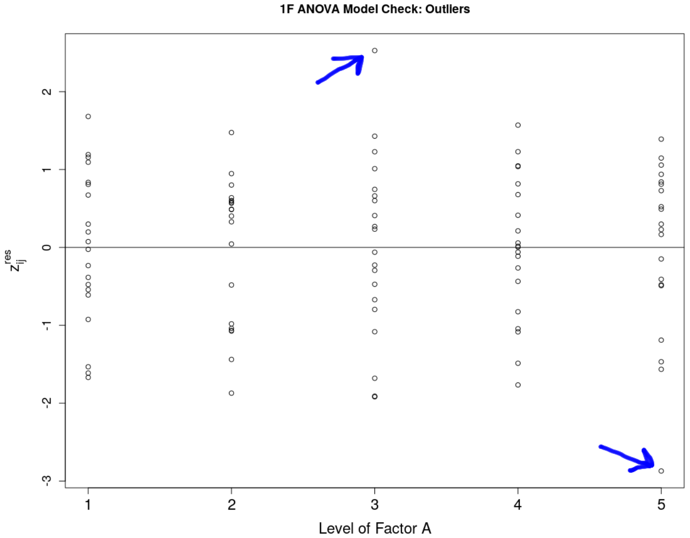
†Dean, Voss *et al*, *Design & Analysis of Experiments*, 2^{nd} Ed, 2017. (§5.2.1)

‡Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.4.1)

ANOVA Model Checking: No Outliers



ANOVA Model Checking: Some (Possible) Outliers



Measurements between two and three std deviations are possibly outliers.
Measurements beyond three standard deviations are definitely outliers.

ANOVA Model Checking: Outlier Mitigation

Q: How to handle outliers when performing 1F ANOVA?

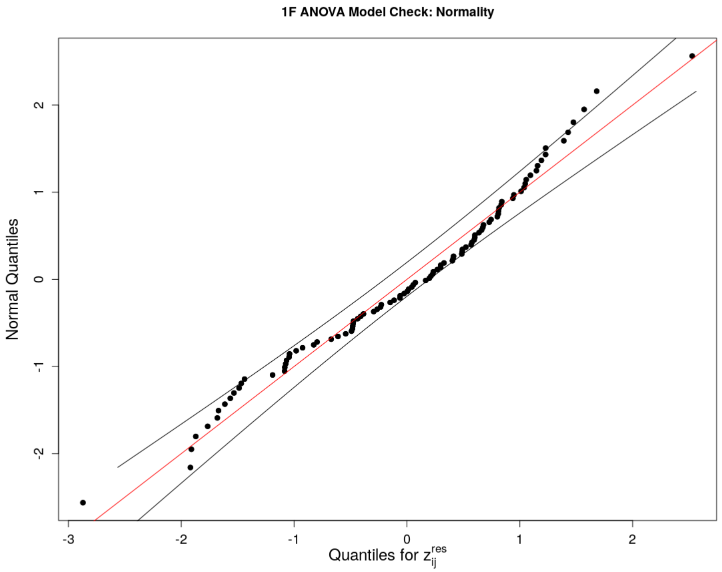
A: For each outlier:

- If outlier was due to measurement/calculation error, correct it[†‡].
- Else, outlier may be due to violation(s) of the ANOVA assumptions[†].
- Else, the 1-factor linear model may be insufficient[†]:
  - Consider building a 2-Factor ANOVA model... (covered in Ch11)
  - ...or an Analysis of Covariance (ANCOVA) model (beyond scope of course)

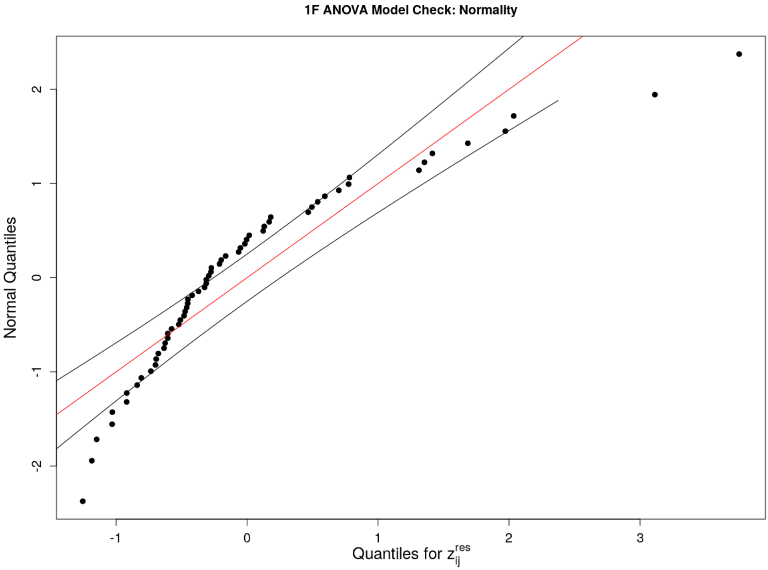
"We should be careful not to reject or discard an outlying observation unless we have reasonably non-statistical grounds for doing so. At worst, you may end up with two analyses; one with the outlier and one without."[‡]

[†]Dean, Voss *et al*, *Design & Analysis of Experiments*, 2^{nd} Ed, 2017. (§5.4)
[‡]Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.4.1)

ANOVA Model Checking: Normality Satisfied



ANOVA Model Checking: Normality Violated



ANOVA Model Checking: Normality Mitigation

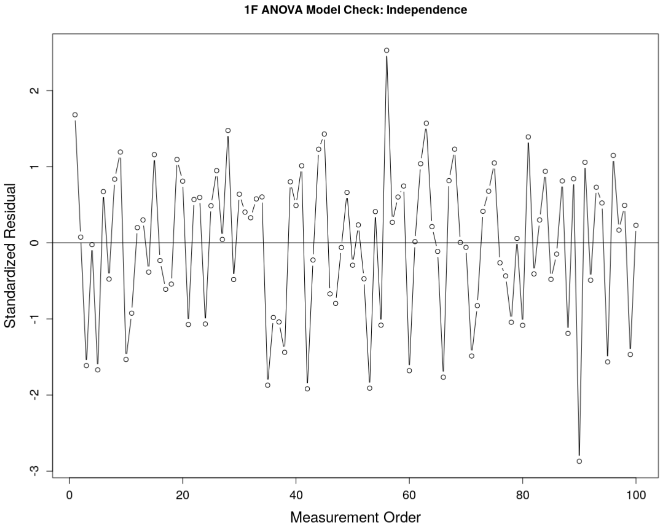
Q: How to perform a 1F ANOVA when the Normality Assumption is violated?

A: Perform a 1F Kruskal-Wallis♠ ANOVA which does not assume normality.

- To be covered in Chapter 15.

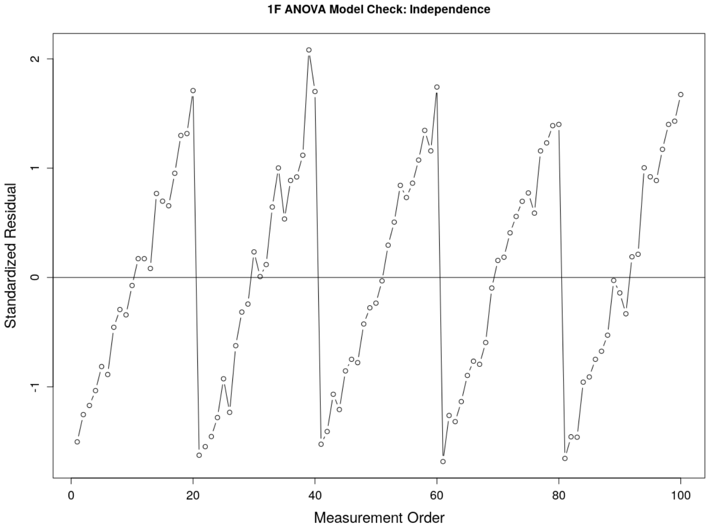
♠W. Kruskal, W. Wallis, "Use of Ranks in One-Criterion Variance Analysis", *Journal of the American Statistical Association*, **47** (1952), 583-621.

ANOVA Model Checking: Independence Satisfied



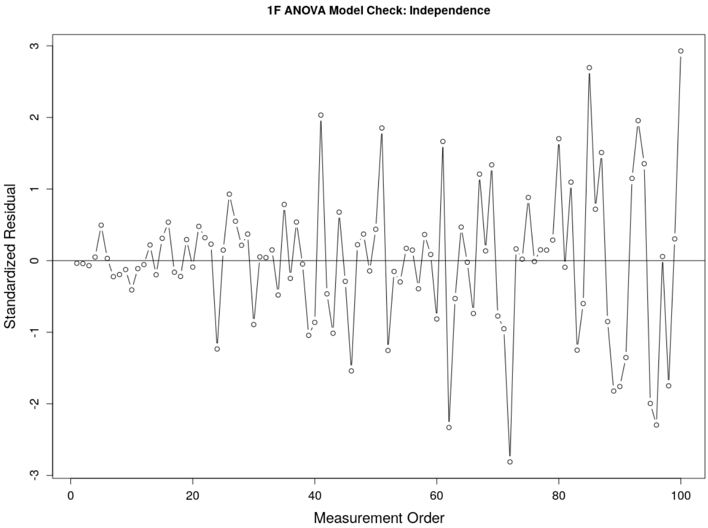
There's no discernible pattern.

ANOVA Model Checking: Independence Violated



There's a clear (cycle) pattern.

ANOVA Model Checking: Independence Violated



There's a clear (fan) pattern.

ANOVA Model Checking: Independence Mitigation

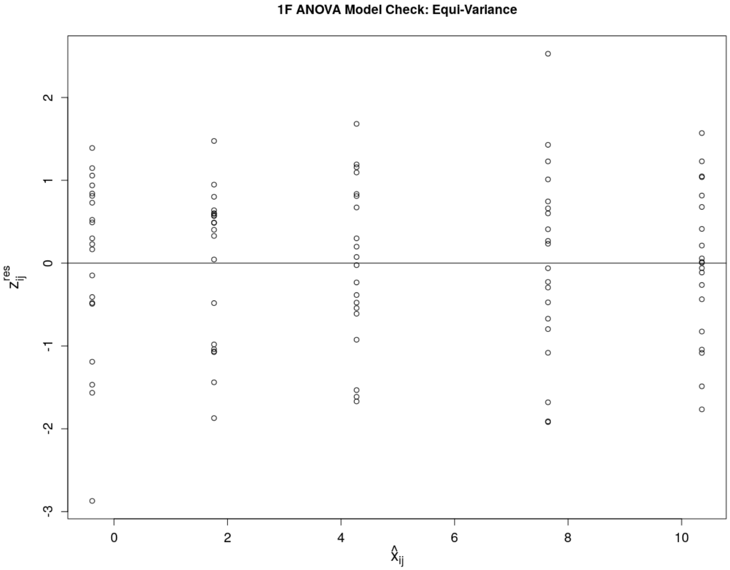
Q: How to perform a 1F ANOVA when the Independence is violated?

A: This is where things become frustrating:

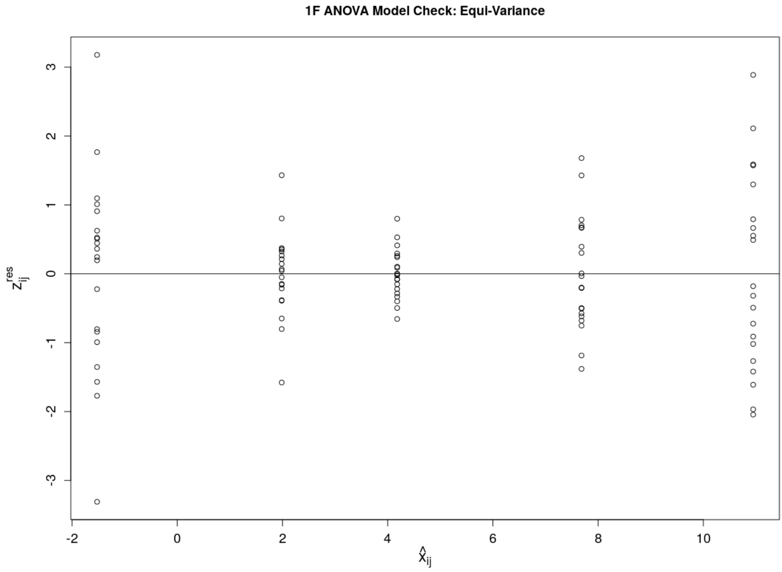
- If randomization was not used, redo the experiment using randomization‡.
- If randomization was used, then use a more complicated model†:
  - 2-Factor ANOVA – to be covered in Ch11
  - Analysis of Covariance (ANCOVA) – beyond scope of this course

†Dean, Voss *et al*, *Design & Analysis of Experiments*, 2^{nd} Ed, 2017. (§5.5)
‡Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, Wiley, 2009. (§3.4.2)

ANOVA Model Checking: Equi-Variance Satisfied



ANOVA Model Checking: Equi-Variance Violated



ANOVA Model Checking: Equi-Variance Mitigation

Q: How to perform 1F ANOVA when Equi-Variance Assumption is violated?

A: Perform an appropriate **variance-stabilizing data transformation**†‡♣ first:

$$\log X,\ \log(1+X),\ \log(1+\min x_{ij}+X),$$
$$\sqrt{X},\ \sqrt{0.5+X},\ \sqrt{X}+\sqrt{1+X},$$
$$1/X,\ 1/\sqrt{X},\ \arcsin(\sqrt{X}),\ 2\arcsin(\sqrt{X\pm 1/2m})$$

If data are counts or Poisson-like, use a square-root transformation†‡♣.
If data are proportions or Binomial-like, use an arcsine transformation†♣.
When in doubt, plot $\log s_i$ vs. $\log(\bar{x}_{i\bullet})$ to help determine data transformation†‡.
If data transformations don't help much, a more robust method is necessary♡.

†Dean, Voss *et al*, *Design & Analysis of Experiments*, 2^{nd} Ed, 2017. (§5.6.2)
‡D.C. Montgomery, *Design & Analysis of Experiments*, 7^{th} Ed, 2009. (§3.4.3)
♣D.C. Howell, *Statistical Methods for Psychology*, 7^{th} Ed, 2010. (§11.9)
♡R.J. Grissom, "Heterogeneity of Variance in Clinical Data", *Journal of Consulting & Clinical Psychology*, **68** (2000), 155-165.

NOTE: Data transformations are beyond the scope of this course.

Textbook Logistics for Section 10.3

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Expected Value	$E(X)$	$\mathbb{E}[X]$
Variance	$V(X)$	$\mathbb{V}[X]$
Sum of Squares of Factor A	SSTr	SS_A
Mean Square of Factor A	MSTr	MS_A
Sum of Squares of Residuals	SSE	SS_{res}
Mean Square of Residuals	MSE	MS_{res}
Effect of i^{th} Factor A	α_i	α_i^A
Null Hypothesis for Factor A	H_0	H_0^A
Alt. Hypothesis for Factor A	H_A	H_A^A

Textbook Logistics for Section 10.3

- Ignore "β for the F Test" section.
  - Used to compute the power of a particular ANOVA.
  - Also used to determine minimum group sizes J_i to ensure a sufficiently-powerful ANOVA.
- Ignore "Relationship of the F Test to the t Test" section.
  - For $I = 2$, the pooled t-test is equivalent to 1-Factor ANOVA.
  - For $I = 2$, the independent t-test is more flexible than 1-Factor ANOVA.
  - For $I > 2$, there's no reliable general test without assuming equal variances.
- Ignore "A Random Effects Model" section.
  - Occurs when the levels of Factor A are chosen for an experiment out of a larger set (or population) of levels rather than choosing all possible levels.
  - The resulting linear model now has random variables $A_i \sim$ Normal$(0, \sigma_A^2)$ for Factor A instead of model parameters $\alpha_i^A \in \mathbb{R}$ such that $\sum_i \alpha_i^A = 0$.
  - The corresponding hypotheses are now: $$H_0^A : \sigma_A^2 = 0$$ $$H_A^A : \sigma_A^2 > 0$$
  - The ANOVA procedure is identical as for fixed effects linear models.
  - However, model assumption checking is subtler and trickier.
  - Also, point estimation of σ^2 is somewhat different.

Fin.