1-Factor Random Effects ANOVA

Engineering Statistics II
Section 10.E

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TTU

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PART I:

1-Factor Balanced Completely Randomized ANOVA
(1F bcrANOVA)

Random Effects Model Assumptions

Random Effects Linear Model

F-Test Procedure

Expected Mean Squares

Point Estimators of σ^2 & σ_A^2

1-Factor ANOVA Random Effects Model Assumptions

Random effects means the levels of factor A are randomly selected.

Proposition

(1F bcrANOVA Random Effects Model Assumptions)

- *(**1 Desired Factor**) Factor A has I levels.*
- *(**Factor Levels are Randomly Selected**) AKA Random Effects.*
- *(**Balanced Replication in Groups**) Each group has $J > 1$ units.*
- *(**Distinct Exp. Units**) All IJ units are distinct from each other.*

- *(**Random Assignment across Groups**)*

- *(**Independence**) All measurements on units are independent.*
- *(**Normality**) All groups are approximately normally distributed.*
- *(**Equal Variances**) All groups have approximately same variance.*

Mnemonic: ***1DF FLaRS BRiG DEU** | **RAaG** | **I.N.EV***

1F bcrANOVA Random Effects Linear Model

Random effects means the levels of factor A are randomly selected.

1F bcrANOVA Random Effects Linear Model

I	$\equiv$	# groups to compare
J	$\equiv$	# measurements in each group
X_{ij}	$\equiv$	rv for j^{th} measurement taken from i^{th} group
μ_i	$\equiv$	Mean of i^{th} population or true average response from i^{th} group
μ	$\equiv$	Common population mean or true average overall response
A_i	$\equiv$	rv for deviation from μ due to i^{th} group
E_{ij}	$\equiv$	Deviation from μ due to random error

ASSUMPTIONS: $E_{ij} \overset{iid}{\sim}$ Normal $\left(0, \sigma^2\right)$, $A_i \overset{iid}{\sim}$ Normal $\left(0, \sigma_A^2\right)$

A_i & E_{ij} are all mutually independent of each other

$$X_{ij} = \mu + A_i + E_{ij}$$

$$H_0^A : \sigma_A^2 = 0$$
$$H_A^A : \sigma_A^2 > 0$$

1F bcrANOVA Random Effects F-Test

1. Determine df's: $n = IJ, \quad \nu_A = I - 1, \quad \nu_{res} = I(J-1)$
2. Compute Group Means: $\bar{x}_{i\bullet} := \frac{1}{J} \sum_j x_{ij}$
3. Compute Grand Mean: $\bar{x}_{\bullet\bullet} = \frac{1}{I} \sum_i \bar{x}_{i\bullet}$
4. Compute $\text{SS}_{res} := \sum_{ij} (x_{ij}^{res})^2 = \sum_i \sum_j (x_{ij} - \bar{x}_{i\bullet})^2$
5. Compute $\text{SS}_A \quad := \sum_{ij} (\hat{\alpha}_i^A)^2 = \sum_i \sum_j (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2$
6. Compute Mean Squares: $\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}}, \quad \text{MS}_A := \frac{\text{SS}_A}{\nu_A}$
7. Compute Test Statistic Value: $f_A = \frac{\text{MS}_A}{\text{MS}_{res}}$
8. Compute F-cutoff/P-value:
   - By hand, lookup $f^*_{\nu_A, \nu_{res}; \alpha}$
   - By SW, compute $p_A = 1 - \Phi_F(f_A; \nu_A, \nu_{res})$
9. Render Decision:
   - If $f_A \geq f^*_{\nu_A, \nu_{res}; \alpha}$, then reject H_0^A; else accept H_0^A.
   - If $p_A \leq \alpha$, then reject H_0^A; else accept H_0^A.

1F bcrANOVA Random Effects (Expected Mean Squares)

Proposition

Given 1F experiment satisfying 1F bcrANOVA random effects assumptions. Then:

$$
\begin{array}{lrcl}
(i) & \mathbb{E}[MS_{res}] & = & \sigma^2 \\
(ii) & \mathbb{E}[MS_A] & = & \sigma^2 + J\sigma_A^2
\end{array}
$$

1F bcrANOVA Random Effects (Point Estimators of σ^2 & σ_A^2)

Proposition

Given 1F experiment satisfying 1F bcrANOVA random effects assumptions. Then:

$$
\begin{aligned}
(i) \quad \hat{\sigma}^2 &= MS_{res} \\
(ii) \quad \hat{\sigma}_A^2 &= (MS_A - MS_{res})/J
\end{aligned}
$$

PART II:

1-Factor Unbalanced Completely Randomized ANOVA
(1F ucrANOVA)

Random Effects Model Assumptions

Random Effects Linear Model

F-Test Procedure

Expected Mean Squares

Point Estimators of σ^2 & σ_A^2

1F ucrANOVA (Random Effects Model Assumptions)

Proposition

(1F ucrANOVA Random Effects Model Assumptions)

- *(**1 Desired Factor**) Factor A has I levels.*
- *(**Factor Levels are Randomly Selected**) AKA Random Effects.*
- *(**Replication in Groups**) Each group has $J_i > 1$ units.*
- *(**Distinct Exp. Units**) All $\sum_i J_i$ units are distinct from each other.*

- *(**Random Assignment across Groups**)*

- *(**Independence**) All measurements on units are independent.*
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Random effects means the levels of factor A are randomly selected.

1F ucrANOVA Random Effects Linear Model

I	$\equiv$	# groups to compare
J_i	$\equiv$	# measurements in i^{th} group
X_{ij}	$\equiv$	rv for j^{th} measurement taken from i^{th} group
μ_i	$\equiv$	Mean of i^{th} population or true average response from i^{th} group
μ	$\equiv$	Common population mean or true average overall response
A_i	$\equiv$	rv for deviation from μ due to i^{th} group
E_{ij}	$\equiv$	Deviation from μ due to random error

ASSUMPTIONS: $E_{ij} \overset{iid}{\sim}$ Normal $\left(0, \sigma^2\right)$, $A_i \overset{iid}{\sim}$ Normal $\left(0, \sigma_A^2\right)$

A_i & E_{ij} are all mutually independent of each other

$$X_{ij} = \mu + A_i + E_{ij}$$

$$H_0^A : \sigma_A^2 = 0$$
$$H_A^A : \sigma_A^2 > 0$$

1F ucrANOVA Random Effects F-Test

1. Determine df's: $n := \sum_i J_i, \quad \nu_A = I - 1, \quad \nu_{res} = n - I$
2. Compute Group Means: $\bar{x}_{i\bullet} := \frac{1}{J_i} \sum_{j=1}^{J_i} x_{ij}$
3. Compute Grand Mean: $\bar{x}_{\bullet\bullet} := \frac{1}{I} \sum_i \bar{x}_{i\bullet}$
4. Compute $\text{SS}_{res} := \sum_i \sum_{j=1}^{J_i} (x_{ij}^{res})^2 = \sum_i \sum_{j=1}^{J_i} (x_{ij} - \bar{x}_{i\bullet})^2$ and $\text{MS}_{res} := \frac{\text{SS}_{res}}{\nu_{res}}$
5. Compute $\text{SS}_A := \sum_i \sum_{j=1}^{J_i} (\hat{\alpha}_i^A)^2 = \sum_i \sum_{j=1}^{J_i} (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2$ and $\text{MS}_A := \frac{\text{SS}_A}{\nu_A}$
6. Compute Test Statistic Value: $f_A = \frac{\text{MS}_A}{\text{MS}_{res}}$
7. Compute P-value: $p_A := \mathbb{P}(F > f_A) \approx 1 - \Phi_F(f_A; \nu_A, \nu_{res})$
8. Render Decision:

(by SW)	If	$p_A \leq \alpha$	then reject H_0^A for H_A^A, else accept H_0^A.
(by hand)	If	$f_A \geq f^*_{\nu_A, \nu_{res}; \alpha}$	then reject H_0^A for H_A^A, else accept H_0^A.

1F ucrANOVA Random Effects (Expected Mean Squares)

Proposition

Given 1F experiment satisfying 1F ucrANOVA random effects assumptions. Then:

$$(i) \quad \mathbb{E}[MS_{res}] = \sigma^2$$

$$(ii) \quad \mathbb{E}[MS_A] = \sigma^2 + \frac{1}{I-1}\left(n - \frac{\sum_i J_i^2}{n}\right)\sigma_A^2, \qquad [n := \sum_i J_i]$$

1F ucrANOVA Random Effects (Point Estimators of σ^2 & σ_A^2)

Proposition

Given 1F experiment satisfying 1F ucrANOVA random effects assumptions. Then:

$$(i) \quad \hat{\sigma}^2 = MS_{res}$$

$$(ii) \quad \hat{\sigma}_A^2 = \frac{MS_A - MS_{res}}{\frac{1}{I-1}\left(n - \frac{\sum_i J_i^2}{n}\right)} \qquad [n := \sum_i J_i]$$

Fin

Fin.